Grades on the Second Exam.

Here is the information on the second test. 16 people took the exam. The scores were 99, 98, 89, 88, 85, 81, 76, 73, 70, 69, 64, 61, 56, 53, 52, and 46. The average was 72.5 with a standard deviation of 16.02. The median was 71.5. The break down in the grades is in the table.

Grade	Range	Number	Percent
A	85–100	5	31.25%
B	75–84	2	12.50%
C	65–74	4	25.00%
D	50–64	4	25.00%
F	0–59	1	6.25%

Mathematics 552 Homework due Friday, March 24, 2006.

Our goal for a while is to use the Cauchy integral formula to deduce as many facts and properties of analytic functions as possable. Let D be a bounded domain with nice boundary and let $f(z)$ be analytic in D and continuous on $D \cap \partial D$. Then the Cauchy integral formula is

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{1}$$

where z is any point in D. If you want to avoid the use of the Greek letter ζ this can be written as

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - a}\, dz$$

We also have forumlas for the first two directives of $f(z)$. These are

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \tag{2}$$

$$f''(z) = \frac{2}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta \tag{3}$$

for $z \in D$. By now you are wondering if there are formulas for the higher derivatives. There are and we now derive them. Hold ζ fixed, let n be a positive integer, and set

$$g(z) = \frac{1}{(\zeta - z)^n} = (\zeta - z)^{-n}.$$

This is analytic (that is complex differentable) at all points $z \neq \zeta$. Therefore for $z \neq \zeta$ we have

$$g'(z) = -n(\zeta - z)^{-n-1}(-1) = n(\zeta - z)^{-n-1} = \frac{n}{(\zeta - z)^{n+1}}.$$

Therefore, by the definition of derivative we, have for $z \neq \zeta$ that

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(\zeta - (z+h))^n} - \frac{1}{(\zeta - z)^n}\right) = \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = g'(z) = \frac{n}{(\zeta - z)^{n+1}}.$$

That is

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(\zeta - (z+h))^n} - \frac{1}{(\zeta - z)^n}\right) = \frac{n}{(\zeta - z)^{n+1}}. \tag{4}$$

Let $H_n(z)$ be the function defined in D by

$$H_n(z) = \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^n}\, d\zeta. \tag{5}$$

Using the limit (4) we have

$$
\begin{aligned}
H_n'(z) &= \lim_{h\to 0}\frac{1}{h}\left(H_n(z+h) - H_n(z)\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\int_{\partial D}\frac{f(\zeta)}{(\zeta-(z+h))^n}\,d\zeta - \int_{\partial D}\frac{f(\zeta)}{(\zeta-z)^n}\,d\zeta\right) \\
&= \lim_{h\to 0}\int_{\partial D}\frac{1}{h}\left(\frac{1}{(\zeta-(z+h))^n} - \frac{1}{(\zeta-z)^n}\right)f(\zeta)\,d\zeta \\
&= \int_{\partial D}\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(\zeta-(z+h))^n} - \frac{1}{(\zeta-z)^n}\right)f(\zeta)\,d\zeta \\
&= \int_{\partial D}\frac{n}{(\zeta-z)^{n+1}}f(\zeta)\,d\zeta \qquad\qquad \text{(Used (4) here)} \\
&= n\int_{\partial D}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta \\
&= nH_{n+1}(z).
\end{aligned}
$$

This shows that the complex derivitive of $H_n(z)$ exist, that is $H_n(z)$ is analytic in D, and its derivative is $nH_{n+1}(z)$. We record this:

Lemma 1. *The for each positive integer n the function $H_n(z)$ defined by* (5) *is analytic in D and its defintive is*

$$H_n'(z) = nH_{n+1}(z).$$

□

Problem 1. Expalin why the Cauchy integral formula can be written as

$$f(z) = \frac{1}{2\pi i}H_1(z).$$

(This is easy, don't make it hard.) □

From Lemma 1 this implies that $f(z)$ has derivative

$$f'(z) = \frac{1}{2\pi i}H_1'(z) = \frac{1}{2\pi i}H_2(z).$$

Problem 2. Show that $f'(z) = \frac{1}{2\pi i}H_2(z)$ is really the same thing as equation (2). (Again this is easy) □

From $f'(z) = \frac{1}{2\pi i}H_2(z)$ and Lemma 1 we have that f' is differentiable, that is analytic, and that for $z \in D$

$$f''(z) = \frac{1}{2\pi i}H_2'(z) = \frac{2}{2\pi i}H_3(z).$$

Problem 3. Show that $f''(z) = \frac{2}{2\pi i}H_3(z)$ is the same as the formula (3) for the second derivative. □

We can keep using Lemma 1 in this fashion:

$$f'''(z) = \frac{2}{2\pi i}H_3'(z) = \frac{2\cdot 3}{2\pi i}H_4(z) = \frac{3!}{2\pi i}H_4(z)$$
$$f^{(4)}(z) = \frac{2\cdot 3}{2\pi i}H_4'(z) = \frac{2\cdot 3\cdot 4}{2\pi i}H_5(z) = \frac{4!}{2\pi i}H_5(z)$$
$$f^{(5)}(z) = \frac{2\cdot 3\cdot 4}{2\pi i}H_5'(z) = \frac{2\cdot 3\cdot 4\cdot 5}{2\pi i}H_6(z) = \frac{5!}{2\pi i}H_6(z)$$
$$f^{(6)}(z) = \frac{2\cdot 3\cdot 4\cdot 5}{2\pi i}H_6'(z) = \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{2\pi i}H_7(z) = \frac{6!}{2\pi i}H_7(z)$$

and so on.

Problem 4. Find the pattern in the above calculations and give a formula for the n-th derivative $f^{(n)}(z)$ of $f(z)$ in D in terms of the functions H_k. Prove your result by use of induction. □

Problem 5. Use your result for the last problem to give an integral formula for $f^{(n)}(z)$. Hint: This should only involve using the definition of H_k. You can check your formula by looking in the text. □