Mathematics 552 Homework due Friday, March 17, 2006.

Problem 1. Use the Cauchy integral formula to compute the following integrals.

(a) $\displaystyle\int_{|z|=3} \frac{e^z}{z}\,dz$

(b) $\displaystyle\int_{|z|=3} \frac{e^z}{z-1}\,dz$

(c) $\displaystyle\int_{|z|=2} \frac{z}{z^2+4z+3}\,dz$ HINT: Factor the denominator. □

Here is the statement we have of the Cauchy integral formula.

Cauchy integral formula. *Let D be a bounded domain with nice boundary ∂D and let $f(z)$ be analytic in D and continuous on $D \cup \partial D$. Then for $z \in D$*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

Problem 2. Know the statement and proof of the Cauchy integral formula. (This will be collected in the from of a quiz.) □

Problem 3. Let D be a bounded domain with nice boundary ∂D and let $f(z)$ be analytic in D and continuous on $D \cup \partial D$. Show that for $z \in D$ the derivative of $f(z)$ can be represented by the integral formula

$$f'(z) = \int_{\partial D} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2}.$$

HINT: By the Cauchy integral formula we have that for $z \in D$ and h small enought that $z + h \in D$ then

$$f(z+h) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{\zeta - z - h}, \qquad f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

Use this to show that

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h} &= \frac{1}{h}\left(\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{\zeta - z - h} - \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{\zeta - z}\right) \\
&= \frac{1}{2\pi i}\int_{\partial D} \frac{1}{h}\left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right) f(\zeta)\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D} \frac{1}{h}\frac{(\zeta - z) - (\zeta - z - h)}{(\zeta - z - h)(\zeta - z)} f(\zeta)\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D} \frac{1}{h}\frac{h}{(\zeta - z - h)(\zeta - z)} f(\zeta)\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D} \frac{1}{(\zeta - z - h)(\zeta - z)} f(\zeta)\,d\zeta
\end{aligned}$$

Therefore

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{1}{2\pi i}\int_{\partial D} \frac{1}{(\zeta - z - h)(\zeta - z)} f(\zeta)\,d\zeta$$

and this can be evaluated by taking the limit under the integral (*i.e.* you can get by with just plugging in $h = 0$). $\square$

Problem 4. Let D be a bounded domain with nice boundary ∂D and let $f(z)$ be analytic in D and continuous on $D \cup \partial D$. Show that for $z \in D$ the second derivative of $f(z)$ exists and can be represented by the integral formula

$$f''(z) = \frac{2}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta. \tag{1}$$

Conclude that $f'(z)$ is analytic in D. HINT: This is very much like the last problem. From that problem we have

$$f'(z+h) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)\, dz}{(\zeta - z - h)^2}, \qquad f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)\, dz}{(\zeta - z)^2}.$$

Use this and do a calculation like the one above to simplify

$$\frac{f'(z+h) - f'(z)}{h} = \frac{1}{h} \left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)\, d\zeta}{(\zeta - z - h)^2} - \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2} \right)$$

to the point that the h in the denominator of $\frac{1}{h}$ cancels out. Then you can compute the limit $\lim_{h \to 0} \frac{f'(z+h) - f'(z)}{h} = f''(z)$ to see that it exists and is given by the formula (1). Now explain, using English, why the existance of the second derivative implies that $f'(z)$ is analytic. $\square$