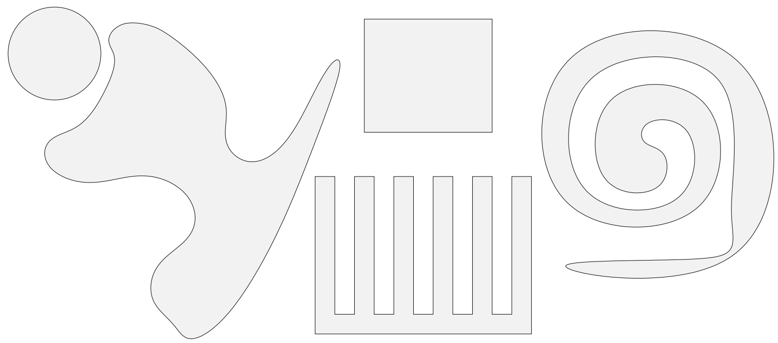
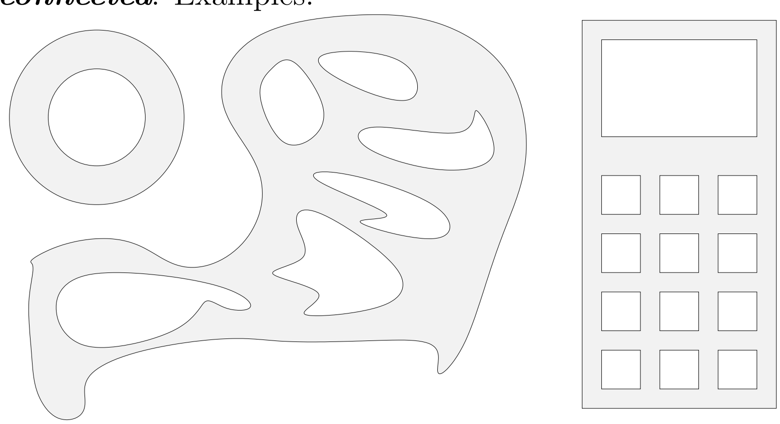
Mathematics 552 Homework due Monday, March 13, 2006.

Recall that a domain is a connected open set. A domain is ***simply connected*** iff it has no holes in it. As examples



Some simply connected domains.

A domain that is not simply connected is called either ***non-simply connected*** or ***multiply connected***. Examples:



Some multiply connected domains.

The official definition high brow definition of simply connected is that any closed curve in the domain can be continuously contracted in the domain to a point. For domains in the plane this is equivalent to the "no holes" definition, and the "no holes" version is easier to visualize.

We have proven

Theorem 1 (A form of Cauchy's Theorem)**.** *If D is a simply connected domain and $f(z)$ is analytic in D, then for any closed curve C in D*

$$\int_C f(z)\,dz = 0.$$

□

We used this to show that an analytic function on a simply connected domain has an antiderivative. More precisely:

Theorem 2 (Exsitance of antiderivatives)**.** *If D is simply and $f(z)$ is analytic in D, then there is an analytic function $F(z)$ on D with $F'(z) = f(z)$.* □

This is false if the domain is not simply connected.

Problem 1. Show that the analytic function $f(z) = 1/z$ does not have an antiderivative on the domain $D := \{z : 1/2 < |z| < 2\}$. Hint: Suppose that $f(z)$

did have any antiderivative $F(z)$. Then for any curve C in D we have $\int_C f(z)\,dz = F(C_{\text{INITIAL}}) - F(C_{\text{END}})$. In particular this implies that if C is a closed curve that $\int_C f(z)\,dz = 0$. Now get a contradiction by showing that if C is the curve $|z| = 1$ transversed counterclockwise that $\int_C f(z)\,dz = \int_C \frac{dz}{z} \neq 0$. □

The existence of antiderivatives has nice consequences.

Theorem 3 (Existance of Logarithms on Simply Connected Domains)**.** *Let D be simply connected and $f(z)$ analytic on D with $f(z) \neq 0$ at any point of D. Then there is an analytic function $h(z)$ in D with $e^{h(z)} = f(z)$. We call $h(z)$ a **logarithm** of $f(z)$.* □

Restatement: A non-vanishing analytic function in a simply connected domain has an analytic logarithm. Note that the function $h(z)$ is not unique. For if $e^{h(z)} = f(z)$ then for any integer n we also have $e^{h(z)+2\pi ni} = f(z)$.

Problem 2. Prove Theorem 3 along the following lines.

(a) Explain (this means using some English) why there is an analytic function $g(z)$ on D with
$$g'(z) = \frac{f'(z)}{f(z)}$$
in D. HINT: Is $f'(z)/f(z)$ analytic in D? (Your can assume that $f'(z)$ is analytic, which we will show later.)

(b) With $g(z)$ as in part (a) show that $e^{-g(z)}f(z)$ is constant. HINT: To show that $e^{-g(z)}f(z)$ is constant it is enough to show that its derivative is zero. Note that
$$\frac{d}{dz}\left(e^{-g(z)}f(z)\right) = -g'(z)e^{-g(z)}f(z) + f'(z)e^{-g(z)}$$
and use that $g'(z) = f'(z)/f(z)$. (At some point you should have a phrase such as "the derivative of **** is identically zero so ***** is constant.)

(c) Because $e^{-g(z)}f(z)$ is constant there is a complex number α with $e^{-g(z)}f(z) = \alpha$. As neither $e^{-g(z)}$ nor $f(z)$ vanish this implies that $\alpha \neq 0$. Therefore there is a complex number β with $e^{\beta} = \alpha$. Thus $e^{-g(z)}f(z) = e^{\beta}$. Explain (again using English) why this implies that $h(z) = \beta + g(z)$ is function we are looking for. □

Now that we have logarithms we can find roots.

Theorem 4 (Existance of roots)**.** *Let D be a simply connected domain and $f(z)$ a function analytic in D with $f(z) \neq 0$ for any $z \in D$. Let n be an a nonzero integer. There there is a analytic function $g(z)$ in D with $g(z)^n = f(z)$. We call $g(z)$ an **n-th root** of $f(z)$.* □

Restatement: Non-vanishing analytic functions on simply connected domains have analytic n-th roots. Note that when $n \neq \pm 1$ that the n-th roots are not unique. For if $g(z)^n = f(z)$, that also $\left(e^{\frac{2\pi ki}{n}} g(z)\right)^n = f(z)$ for any integer k.

Problem 3. Prove Theorem 4 along the following lines.

(a) Use Theorem 3 to find a function $h(z)$ with $e^{h(z)} = f(z)$.

(b) Let $g(z) = e^{\frac{1}{n}h(z)}$ and explain why $g(z)$ is the function we want. □