Mathematics 552 Homework due Wednesday, March 3, 2006.

Recall that if $f(x, y)$ is a function of two variables, then its ***gradient*** is the vector field $\nabla f(x, y) = (f_x, f_y)$. A standard fact is that the gradient is perpendicular to the curves defined by $f(x, y) = C$ where C is a constant.

Problem 1. Let $f = u + iv$ be an analytic function in an domain U.

(a) Use the Cauchy-Riemann equation so show that at each point of U that $\|\nabla u\| = \|\nabla v\|$ (that is at any point of U the gradients of u have the same length) and that ∇u and ∇v are always perpendicular. (That is the dot product $\nabla u \cdot \nabla v = 0$.)

(b) Use that ∇u and ∇v are always perpendicular to explain why for any constants a and b the curves $u = a$ and $v = b$ meet at right angles. (At least if the curves meet at a point where $f'(z) \neq 0$.)

(c) Let $f(z) = z^2$. Find u and v and graph some of the curves $u = a$ and $v = b$. □

Shortly we will need to know how to expand some ratioanal function into series. Recall that if w is a complex number with $|w| < 1$ that

$$\frac{1}{1-w} = 1 + w + w^2 + w^3 + \cdots = \sum_{k=0}^{\infty} w^k. \tag{1}$$

Problem 2. If z, z_0, and ζ are complex numbers with $|z - z_0| < |\zeta - z_0|$ show that

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0)}\left(1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \left(\frac{z - z_0}{\zeta - z_0}\right)^3 + \cdots\right) \\ &= \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}.\end{aligned}$$

Hint: Start with

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \left(\frac{1}{\zeta - z_0}\right)\left(\frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}\right)$$

and use (1). □

Problem 3. If z, z_0, and ζ are complex numbers with $|\zeta - z_0| < |z - z_0|$ show that

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{-1}{(z - z_0)}\left(1 + \left(\frac{\zeta - z_0}{z - z_0}\right) + \left(\frac{\zeta - z_0}{z - z_0}\right)^2 + \left(\frac{\zeta - z_0}{z - z_0}\right)^3 + \cdots\right) \\ &= -\sum_{k=1}^{\infty} \frac{(\zeta - z_0)^{k-1}}{(z - z_0)^k}.\end{aligned}$$

Hint: This time start with

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \left(\frac{-1}{z - z_0}\right)\left(\frac{1}{1 - \left(\frac{\zeta - z_0}{z - z_0}\right)}\right)$$

and use (1). □