Mathematics 552 Homework due Wednesday, March 1, 2006

Problem 1. Let $f(z), g(z)$ be a continuous complex valued functions defined on curve C parameterized by $z(t)$ with $a \le t \le b$.

(a) Show $\displaystyle\int_C (f(z)+g(z))\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

(b) Show that for any complex number α $\displaystyle\int_C \alpha f(z)\,dz = \alpha \int_C f(z)\,dz$ □

We have shown that if $f(z)$ is analytic in a domain D and has an antiderivative $F(z)$ in D, that if $F'(z) = f(z)$. Then for any curve C in D

$$\int_C f(z)\,dz = F(C_{\text{END}}) - F(C_{\text{INITIAL}})$$

where C_{INITIAL} is the initial point of C and C_{END} is the end point of C.

Problem 2. Use this to compute the following integrals.

(a) $\displaystyle\int_C z^{11}\,dz$ where C is the line segment from 0 to $1+i$. Simplify your answer.

(b) $\displaystyle\int_C \cos(z)\,dz$ where C is the top half of the circle $|z| = \pi/2$ going from $-\pi/2$ to $\pi/2$.

(c) $\int_C z^3 e^{z^4}\,dz$ where C is the circle $|z-3| = 5$ transversed once in the counterclockwise direction. Hint: Before doing any computations thank about what how the initial and end points of C are related. □

Problem 3. This problem is in part to review a bit about the functions x^α. Let U be the open set $U \setminus \{z : \operatorname{Im} z = 0, \operatorname{Re} z < 0\}$. That is U is the complex plane with the nonnegative real numbers deleted. Let α be a complex number and let

$$f(z) = \text{Principle branch of } z^\alpha.$$

Explicitly $f(z)$ is the function defined in U by

$$f(z) = e^{\alpha \operatorname{Log}(z)} \tag{1}$$

where $\operatorname{Log}(z)$ is the principle branch of the logarithm. We have shown that $\operatorname{Log}(z)$ is analytic in U with derivative $1/z$.

(a) Draw U and explain briefly why $f(z)$ is analytic in U.

(b) If $\alpha \ne -1$ find an antiderivative for $f(z)$. That is we what a function $F(z)$ with $F'(z) = f(z)$. Give precise definition of $F(z)$ and explain why it works (this will involve at least a bit of English). Hint: What to you expect the antiderivative of z^α to be? Now define this along the lines of equation (1) and verify that if works.