Mathematics 552 Homework due Friday, February 24, 2006

Recall that a map $f\colon U_1 \to U_2$ between two domains is ***conformal*** iff it is one to one, onto, and analytic.

Problem 1. (a) Draw $U_1 = \{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$. (The first quadrant.)
(b) Draw $U_2 = \{z : \mathrm{Im}(z) > 0\}$. (The upper half plane.)
(c) Show that $f(z) = z^2$ is a conformal map from U_1 to U_2.

Problem 2. Let $0 < \alpha < 2\pi$.
(a) Draw $U_1 = \{z : 0 < \mathrm{Im}(z) < \alpha\}$.
(b) Draw $U_2 = \{z : 0 < \mathrm{Arg}(z) < \alpha\}$.
(c) Show that $f(z) = e^z$ is a conformal map from U_1 to U_2.

The following is a bit trickier.

Problem 3. Let $D = \{z : |z| < 1\}$ be the open unit disk. Let $|a| < 1$ and let φ be the Möbius transformation

$$\varphi(z) = \frac{z-a}{\overline{a}z - 1}.$$

Show that φ is a conformal map of D to itself.

Problem 4. If C is the line segment from $1+2i$ to $3+4i$ this compute

(a) $\displaystyle\int_C z^2\,dz.$ (b) $\displaystyle\int_C \overline{z}\,dz$

Problem 5. Let $r_0 > 0$ and C the curve parameterized by $z(t) = r_0 e^{it}$ with $0 \le t \le 2\pi$. (This parameterizes the circle $|z| = r_0$ in the counterclockwise direction.) Compute

(a) $\displaystyle\int_C z^2\,dz.$ (b) $\displaystyle\int_C \frac{dz}{z}.$

Problem 6. Let c be the complex number $c = a + bi$. Then the antiderivative of $f(z) = e^{cz}$ is $F(z) = \dfrac{e^{cz}}{c}$. By taking real and imaginary parts of thus find the real integrals

$$\int e^{ax}\cos(bx)\,dx, \qquad \text{and} \qquad \int e^{ax}\sin(bx)\,dx.$$

Problem 7. Let D be a bounded domain with nice boundary and $f(z) = u + iv$ and analytic function on D. Then apply Green's theorem (see below) to the integral

$$\int_{\partial D} f(z)\,dz$$

and use the Cauchy-Riemann equations to simplify.

Quiz on Friday.

Know the statement of Green's theorem.

Green's Theorem. *If D is a bounded domain in $\mathbf{R}^2$ with nice boundary ∂D, and $P(x,y)$ and $Q(x,y)$ have continuous first partial derivatives, then*

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(-P_y + Q_x\right)\,dxdy.$$