Mathematics 552 Homework due Wednesday, February 22, 2006

Let α and z be complex numbers with $z \neq 0$. Then set

$$z^\alpha := e^{\alpha \log(z)} \tag{1}$$

where, as usual,

$$\log(z) = \ln(|z|) + i \arg(z).$$

Note in general z^α will be multivalued because log is multivalued. But let's see what happens for some special values of α. First we try $\alpha = 1$.

$$z^1 = e^{1 \log(z)} = e^{\log(z)} = z$$

as we would like. For $\alpha = 2$, and using that $e^{2\pi i} = 1$,

$$\begin{aligned} z^2 &= e^{2\log(z)} = e^{2(\ln(|z|)+i\,\mathrm{Arg}(z)+2n\pi i)} = e^{2(\ln(|z|)+i\,\mathrm{Arg}(z))+4n\pi i} \\ &= e^{2(\ln(|z|)+i\,\mathrm{Arg}(z))}e^{4n\pi i} = \left(e^{\ln(|z|)+i\,\mathrm{Arg}(z)}\right)^2 (1) \\ &= (z)^2 = z^2 \end{aligned}$$

where the first z^2 is as defined by (1) and the last z^2 is the sense of $z^2 = z \cdot z$ that are are use to. Therefore the new definition of z^2 agree with the familiar one and is single valued. Likewise

$$\begin{aligned} z^{-2} &= e^{-2\log(z)} = e^{-2(\ln(|z|)+i\,\mathrm{Arg}(z)+2n\pi i)} = e^{-2(\ln(|z|)+i\,\mathrm{Arg}(z))-4n\pi i} \\ &= e^{-2(\ln(|z|)+i\,\mathrm{Arg}(z))}e^{-4n\pi i} = \left(e^{\ln(|z|)+i\,\mathrm{Arg}(z)}\right)^{-2} (1) \\ &= (z)^{-2} = z^{-2} \end{aligned}$$

This for $\alpha = -2$ the new definition of z^{-2} agrees with the old one and is single valued.

Problem 1. If α is an integer, positive or negative, show that z^α is single valued and the new definition of z^α agrees with the old high school algebra of z^α. □

Now let $\alpha = \dfrac{2}{3}$. Then

$$\begin{aligned} z^{\frac{2}{3}} &= e^{\frac{2}{3}\log(z)} = e^{\frac{2}{3}(\ln(|z|)+i\,\mathrm{Arg}(z)+2n\pi i)} = e^{\frac{2}{3}(\ln(|z|)+i\,\mathrm{Arg}(z))+\frac{4n}{3}\pi i} \\ &= e^{\frac{2}{3}(\ln(|z|)+i\,\mathrm{Arg}(z))}e^{\frac{4n}{3}\pi i} \end{aligned}$$

The complex number $e^{\frac{2}{3}(\ln(|z|)+i\,\mathrm{Arg}(z))}$ is well defined (i.e. single valued), but as n varies over the integers $e^{\frac{4n}{3}\pi i}$ varies over the three cube roots of 1 and therefore $z^{\frac{2}{3}}$ is three valued. More generally

Problem 2. Show that if $\alpha = \frac{p}{q}$ is a rational number in lowest terms, then z^α is a q-valued. That is for each value of $z \neq 0$ there are q values of z^α. □

This leaves one case.

Problem 3. If α is not a rational number show that z^α always has in infinite number of values. □.

To get a single valued version of z^α we define

$$\text{Principle branch of } z^\alpha := e^{\alpha\,\mathrm{Log}(z)}.$$

This has the usual defect of having a jump discounted along the negative real axis.

Problem 4. Compute the following.

(a) $1^{\sqrt{2}}$.
(b) $(2i)^i$.
(c) The Principle branch of $(2i)^i$.
(d) $16^{\frac{3}{4}}$ and put the answers in the form $a + bi$. □

Problem 5. Read pages 68–73 (up to Green's formula) about line integrals in the text and compute the following. (cf. Example 3.1.3 on page 73 of the text).

(a) $\int_C (x^2 + y^2)\,dx + 3xy\,dy$ where C is the curve $y = x^2$ from the point $(0,0)$ to $(1,1)$.
(b) $\int_C (x^2 + y^2)\,dx + 3xy\,dy$ where C is the straight line segment from the point $(0,0)$ to $(1,1)$.
(c) $\int_C (x^2 + y^2)\,dx + 3xy\,dy$ where C is the curve $z(t) = t^2 + it^3$ from the point $(0,0)$ to $(1,1)$.

Computing π.

Josh ask the question of using the series

$$\begin{aligned}\arctan(r) &= \int_0^r \frac{dx}{1+x^2} = \int_0^r (1 - x^2 + x^4 - x^6 + \cdots)\,dx \\ &= r - \frac{r^3}{3} + \frac{r^5}{5} - \frac{r^7}{7} + \frac{r^9}{9} - \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{2k+1}.\end{aligned}$$

Historically this is how this was used. From the addition formula for tan we can show

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

That is

$$\pi = 16\arctan\left(\frac{1}{5}\right) - 4\arctan\left(\frac{1}{239}\right).$$

This leads to the series

$$\pi = 16\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)5^{2k+1}} - 4\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(239)^{2k+1}}$$

This was used by Shanks in 1873 to compute π to 707 decimal places. However it was found in 1945 that he was wrong after the 527-th decimal place.

To the best of my knowledge (as supplemented by Google) the current record is 1,241,100,000,000 digits by Kanada, Ushiro, and Kurodo in 2002.