Mathematics 552 Homework due Monday, February 20, 2006

We continue with our project of extending the definitions of some familiar functions of a real variable to functions of a complex variable. Section 2.5, pages 52–62 of the text has a good presentation of this material. First we show a well known property of the real exponential also holds for the complex version.

Problem 1. Show that e^z is never zero. HINT: Use that $e^z \cdot e^{-z} = e^0 = 1$. □

We also show that the zeros of cos and sin are just the usual real zeros.

Problem 2. Show that the zeros of $\cos(z)$ are $\pi/2 + n\pi$ and the zeros of $\sin(z)$ are $n\pi$ where n varies over the integers. HINT: Rather than do this from scratch, use that we have already shown for $z = x + iy$, that $|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$ and $|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$. You may use that when y is real that $\sinh(y) = \frac{1}{2}(e^y - e^{-y})$ only vanishes when $y = 0$. □

We have already seen that $e^{2\pi i} = 1$ and $e^{\pi i} = -1$. This imply

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(1) = e^z, \qquad e^{z+\pi i} = e^z e^{\pi i} = e^z(-1) = -e^z.$$

Use these facts to show

Problem 3. Show that $\cos(z + 2\pi) = \cos(z)$, $\sin(z + 2\pi) = \sin(z)$, $\cos(z + \pi) = -\cos(z)$, and $\sin(z + \pi) = -\sin(z)$.

We now extend the definition of tan, sec, csc from real values to complex values in the natural way.

$$\tan(z) = \frac{\sin z}{\cos(z)}, \quad \cot(z) = \frac{\cos(z)}{\sin(z)}, \quad \sec(z) = \frac{1}{\cos(z)}, \quad \csc(z) = \frac{1}{\sin(z)}.$$

With these definitions all the usual identities hold. For practice:

Problem 4. Use the properties of we have already shown about $\sin(z)$ and $\cos(z)$ to show

(a) $\tan(z + \pi) = \tan(z)$,
(b) $\dfrac{d}{dz}\tan(z) = \sec^2(z)$,
(c) $1 + \tan^2(z) = \sec^2(z)$.

We now get to a messier function, the logarithm. First we need to think some more about the argument function arg. Our definition of this is if $z = 0$, the $\arg(z)$ is undefined, and if $z \neq 0$, then write z is polar form $z = re^{i\theta}$ and

$$\arg(z) = \arg(re^{i\theta}) = \theta + 2n\pi$$

where n varies over the integers. We would like to have a single valued version of arg. Thus for $z \neq 0$ define $\operatorname{Arg}(z)$ by

$$\operatorname{Arg}(z) = \theta \quad \text{where} \quad z = re^{i\theta} \quad \text{and} \quad -\pi < \theta \leq \pi.$$

That is $\operatorname{Arg}(z)$ is the value of $\arg(z)$ that satisfies $-\pi < \operatorname{Arg}(z) \leq \pi$. We call Arg the ***principle value*** or ***principle branch*** of arg. While Arg is single valued, it has the defect that it is not continuous along the negative real axis where it has a jump discontinuity.

Back to the logarithm. We want the logarithm to be the inverse of the exponential. Let $z \neq 0$ and let

$$z = re^{i\theta} = |z|e^{i\arg(z)}$$

be the polar form of z. Set $w = \ln|z| + i\theta$

$$e^w = e^{\ln|z|+i\theta} = e^{\ln|z|}e^{i\theta} = |z|e^{i\theta} = z.$$

This motivates the definition of $\log(z)$ as

$$\log(z) := \ln(|z|) + i\arg(z).$$

This has the basic property that we wand, that is for $z \neq 0$,

$$e^{\log(z)} = z.$$

It does have one disadvantage, that it is multivalued (because arg is multivalued). So we define the ***principle branch*** of the logarithm by

$$\mathrm{Log}(z) = \ln(|z|) + i\,\mathrm{Arg}(z).$$

This is single valued, but it is has a jump discontinuity along the negative real axis.

We would like log to be analytic on the set $z \neq 0$. Let $z = x + iy$. Then

$$\begin{aligned}\log(z) &= \ln(|z|) + i\arg(z) = \ln(\sqrt{x^2+y^2}) + i\arg(x+iy)\\ &= \frac{1}{2}\ln(x^2+y^2) + i\arctan\frac{y}{x} + ic\end{aligned}$$

where the constant c depends on what quadrant z is located in.

Problem 5. The calculation above shows that $\log(z) = u+iv$ where $u = \frac{1}{2}\ln(x^2+y^2)$ and $v = \arctan\frac{y}{x}+c$. Show that these functions satisfy the Cauchy-Riemann equations and thus $\log(z)$ is analytic on $\{z \neq 0\}$. □

Problem 6. We know that if a function $f(z) = u+iv$ is analytic, then its derivative is given by $f'(z) = u_x + iv_x$. Use this to show that

$$\frac{d}{dz}\log(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}.$$

Therefore the derivative of $\ln(z)$ is what it should be. □

Grades on the First Exam.

Here is the information on the first test. 18 people took the exam. The scores were 106, 94, 91, 88, 87, 77, 76, 71, 63, 62, 55, 48, 48, 41, 41, 41, 40, and 38. The average was 64.83 with a standard deviation of 22.08. The median was 62.5. The break down in the grades is in the table.

Grade	Range	Number	Percent
A	90–100	3	16.67%
B	80–89	2	11.11%
C	70–79	3	16.67%
D	60–69	2	11.11%
F	0–59	8	44.44%

Warning.

This course starts out with the easiest material first, so this exam was the easiest one of the term. Therefore if you did not do well on it you are in trouble. The last day to drop the class without getting a WF is Monday, February 20. Judging from what I have seen in past classes, ***anyone who got below 50 on this exam is very likely better off dropping the course now.***