Mathematics 552 Homework due Monday, February 13, 2006

Reminder: The first test is Wednesday, February 15.

We have defined a ***linear fractional transformation*** or ***Möbius transformation*** to be a transformation of the form

$$f(z) = \frac{az+b}{cz+d} \qquad \text{where} \qquad ad - bc \neq 0.$$

See the text pp. 43–51 for a nice discussion of these transformations.

(1) Show that if $f(z)$ and $g(z)$ are both Möbius transformations then so is the composition $f \circ g(z) = f(g(z))$.

If $f(z) = \dfrac{az+b}{cz+d}$ is a Möbius transformation we define

$$f(\infty) = \frac{a}{c}$$

and

$$f(-d/c) = \infty.$$

(The number $z = -d/c$ is the number that makes the denominator equal to zero.)

(2) With these conventions show that for any three distinct complex numbers z_1, z_2, z_3 that there is a unique Möbius transformation with

$$f(z_1) = 0, \qquad f(z_2) = 1, \qquad f(z_3) = \infty.$$

(3) Find the Möbius transformation $f(z)$ such that $f(2) = 0$, $f(3) = 1$, and $f(i) = \infty$.

(4) Find a Möbius transformation that maps the unit disk $\{|z| < 1\}$ onto upper half space $\{\operatorname{Im} z > 0\}$. Hint: Such a transformation will map the unit circle $\{|z| = 1\}$ onto the real line $\{\operatorname{Im} z = 0\}$. So try mapping three points on the unit circle , say 0, 1, i to the three points on the real line, say 0, 1 and ∞. Then check to see if you have mapped upper rather than the lower half plane. If your mapped onto the lower half plane just multiply what you have by -1.

We now return to the Cauchy-Riemann equations. Recall that these are that if $f(z) = u(x,y) + iv(x,y)$ is analytic then

$$u_x = v_y, \qquad u_y = -v_x.$$

A function $h(x,y)$ is ***harmonic*** iff it satisfies the differential equation

$$h_{xx} + h_{yy} = 0.$$

(5) Use the Cauchy-Riemann equations to show that if $f(z) = u(x,y) + iv(x,y)$ is analytic, then both u and v are harmonic.

We will need the chain rule for functions of two variables. Let $h(x, y)$ be a differentiable function of x and y and assume that $x = x(t)$ and $y = y(t)$ are differentiable functions of t. Then the chain rule is

$$\frac{d}{dt}h(x(t), y(t)) = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}.$$

If we use the physicists convection that $\dot{x} = \dfrac{dx}{dt}$ this is

$$\frac{d}{dt}h(x(t), y(t)) = h_x\dot{x} + h_y\dot{y}.$$

Quiz on Monday. Know the following chain rule just given and also that if $f(z) = u + iv$ is analytic that

$$f'(z) = u_x + iv_x = v_y - iu_y.$$