# Mathematics 552 Final take home version. 2006.

## Due Thursday, April 27, 2:00 p.m.

***You may talk with each other about the problems, but this does not include copying each other. The problems should be written up neatly and using liberal amounts of English in your explanations.***

This will cover a left over topics that we did get to in class. This is Rouche's Theorem on the number of roots of an analytic function in a domain. This is based on the following calculus result.

**Theorem 1.** *Let $\varphi\colon [a,b] \to \mathbf{C}$ be a continuous function that only takes on integer values. Then $\varphi$ is constant.*

*Proof.* We do not give a complete proof, but give the geometric idea behind it. The basic idea is that a continuous function is one where the graph can be drawn without lifting the pencil off the paper. But if the function only takes values in the integers as is non-constant, then at some point $t_0$ in drawing the graph we have to jump from

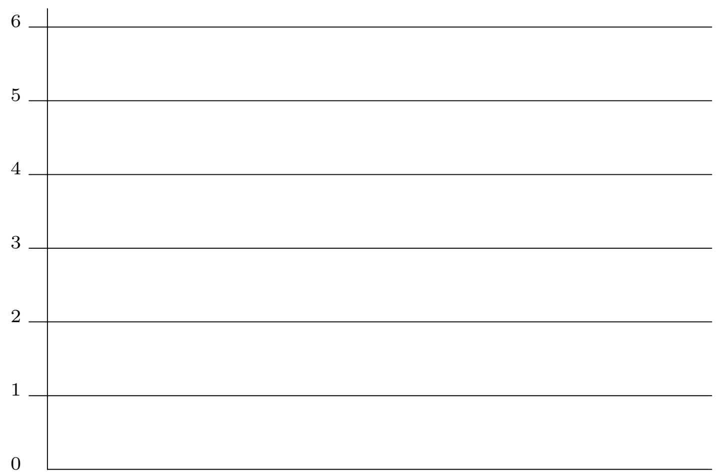


FIGURE 1. An function that take values in the integers, which means that its graph will be entirely on the horizontal lines, must either be constant, or have a jump discontinuity where it jumps from one line to another.

one integer value to another (see Figure 1). But then at $t_0$ there will be a jump discontinuity at $t_0$ contradicting that $\varphi$ is continuous. □

You proved the following on your last homework.

**Theorem 2.** *Let $D$ be a bounded domain with nice boundary. Let $f(z)$ be analytic in $D$ and continuous in $D \cup \partial D$. Assume that $f(z)$ has no zeros on $\partial D$. Then*

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}\,dz = \textit{Number of zeros of } f(z) \textit{ in } D$$

*where the number of zeros is counted with mutiplicity. (That is a zero of order $k$ counts $k$ times.)* □

**Lemma 3.** *If $a$ and $b$ are complex numbers, then $|a+b| \ge |a| - |b|$.*

**Problem 1** (5 points)**.** Prove Lemma 3. □

**Theorem 4** (Rouche's Theorem)**.** *Let $D$ be a bounded domain with nice boundary. Let $f(z)$ and $g(z)$ be analytic in $D$ and continuous in $D \cup \partial D$. Assume that $f(z)$ has no zeros on $\partial D$ and that*

$$|g(z)| < |f(z)| \quad \textit{For all } z \in \partial D.$$

*Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros in $D$ (again counted with multiplicity).*

**Problem 2** (15 points)**.** Prove Rouche's Theorem by doing the following.

(a) Let $0 \le t \le 1$. Then show for $z \in \partial D$ that $|f(z) + tg(z)| \ge |f(z)| - |g(z)|$. HINT: Use Lemma 3 to get that $|f(z)+tg(z)| \ge |f(z)| - |tg(z)|$, and then use that $|t| \le 1$.

(b) Show that if $0 \le t \le 1$ and $z \in \partial D$ that $|f(z) + tg(z)| > 0$.

(c) Define a function $\varphi\colon [0,1] \to \mathbf{C}$ by

$$\varphi(t) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\, dz.$$

Explain why $\varphi$ is constant. HINT: For $t \in [0,1]$ let $h(z) = f(z) + tg(z)$. This is an analytic function in $D$ that is continuous on $D \cup \partial D$ and by part (b) of this problem $h(z)$ has no zero on $\partial D$. Therefore

$$\text{Number of zeros of } h(z) \text{ in } D = \frac{1}{2\pi i} \int_{\partial D} \frac{h'(z)}{h(z)}\, dz.$$

Now explain why $\frac{1}{2\pi i} \int_{\partial D} \frac{h'(z)}{h(z)}\, dz = \varphi(t)$ and why this shows that $\varphi$ only takes on integer values. But as $f(z) + tg(z) \ne 0$ for all $z \in \partial D$ and $t \in [0,1]$ this implies that $\varphi$ is continuous (you can assume this). Now use Theorem 1.

(d) As $\varphi(t)$ is constant, this implies that $\varphi(0) = \varphi(1)$. Explain why this, combined with Theorem 2, completes the proof of Rouche's Theorem. □

*Example.* Here is an example of the use of Rouche's Theorem. Show that $p(z) = 5z^4 + 2z^3 + z - 1$ has exactly four roots in $D = \{z : |z| < 1\}$. **Solution:** Let $f(z) = 5z^4$ and $g(z) = 2z^3 + z - 1$. Then $p(z) = f(z) + g(z)$. The roots of $f(z)$, that is the solutions of $f(z) = 5z^4 = 0$, are $z = 0, 0, 0, 0$ (reminber we count roots with multiplicty). Therefore $f(z)$ has exactly four zeros in $D$ (counting with mutiplicity). The boundary $\partial D$ is the set of $z$ with $|z| = 1$. Therefore on $\partial D$ we have

$$|f(z)| = |5z^4| = 5|z|^4 = 5 \qquad (\text{as } |z| = 1 \text{ on } \partial D).$$

and

$$|g(z)| = |2z^3 + z - 1| \le 2|z|^3 + |z| + |-1| = 2 + 1 + 1 = 4 \qquad (\text{as } |z| = 1 \text{ on } \partial D).$$

Therefore $|g(z)| \le 4 < 5 = |f(z)|$ on $\partial D$. So by Rouche's Theorem $f(z)$ and $f(z) + g(z)$ have the same number of roots in $D$. Thus $5z^4 + 2z^3 + z - 1 = f(z) + g(z)$ has four roots in $D$. □

**Problem 3** (5 points)**.** Show that $p(z) = 7z^6 + z^4 + 2z^3 + z^2 - z + 1$ has exactly seven roots in $D = \{z : |z| < 1\}$. HINT: Use $f(z) = 7z^6$ and $g(z) = z^4 + 2z^3 + z^2 - z + 1$ in Rouche's Theorem. □

**Problem 4** (5 points)**.** Show that $p(z) = z^9 + 5z^4 + 2z^3 + 1$ has exactly four roots in $D = \{z : |z| < 1\}$. HINT: Use $f(z) = 5z^4$ and $g(z) = z^9 + 2z^3 + 1$ in Rouche's Theorem. □