Mathematics 552 Test #3

Take Home Exam: This is due in class on Friday April 23. If you are not going to be in class that day make sure that you get me your exam before then (either in my mail box in the mathematics department, or in the holder on my office door (LC 304)). You may work together on this test, but of course this does not mean just copying an answer that someone has done. Here are two methods of collaboration that I have found useful.

1. Start on a problem together and work on it jointly in a small group (two or three, five is getting to be too large). I find this easiest at a black board (no spectators, everyone standing at the board). Others find it easier working a table working on a pad in such a way that everyone can see what is going on.
2. Work on a problem separately and then compare answers and methods. Make sure that each person involved understands what the others are going and why it is right or wrong.

As you have several days to do the exam I expect the results of be well written and complete. ***Complete means more English, not necessarily more algebra.*** It will not bother me if you say "it now follows by a calculation that" and skip a little of the algebra. (For example something like "Then we have $\lambda^2 - 2\lambda + 2 = 0$ and so by the quadratic formula it follows that $\lambda = 1 \pm i$" is fine.) But you should explain what you are computing and what it has to do with the solution to the problem. One good way to do this is to exchange first drafts of a problem with someone else that has done the problem and have them make comments on where they had trouble reading your solution while you make a similar comments on their paper. You don't have to end up agreeing what is the "best" way to write things, but if someone is having trouble reading what you have written then it probably means that you needed to do some rewriting.

1. (10 Points) Explain why if $p(z)$ is a polynomial then on any closed contour Γ we have

$$\oint_\Gamma p(z)\, dz = 0.$$

(HINT: Do polynomials have antiderivatives?)

2. (10 Points) Let D be a domain with boundary Γ transversed so that D is always on the left. Then show that

$$\int_\Gamma \overline{z}\,dz = 2i\,\mathrm{Area}(D)$$

(HINT: Green's theorem.)

3. (20 Points) Show that if the function f is continuous on the directed smooth curve γ and bounded by the constant M on γ (that is $|f(z)| \le M$ for all z on γ) then

$$\left|\int_\gamma f(z)\,dz\right| \le M\,\mathrm{Length}(\gamma).$$

(HINT: We outlined one approach to this in class based on the arclength formula for a curve. For another approach, based on Riemann sums, see page 121 of the text.)

4. (20 Points.) Problem 16 page 151 of the text.
5. (20 Points) Give all the details in the proof of **Liouville's Theorem:** *A bounded entire function is constant.* (HINT: This is Theorem 21 on page 163 of the text, where the proof is a bit on the short side. Fill in the details.)
6. (20 Points) Let D be a domain, f a function that is analytic in D and its boundary Γ. Assume that f is bounded by M in D and that $z_0 \in D$. Set $R = \mathrm{dist}(z_0, \Gamma)$. Then show that the derivatives of f at z_0 satisfy

$$|f^{(n)}(z_0)| \le \frac{n!\,M}{R^n}.$$

(HINT: Theorem 20 on page 162 of the text.)