More information about line and contour integrals.

Real line integrals. Let $P(x,y)$ and $Q(x,y)$ be continuous real valued functions on the plane and let γ be a smooth directed curve. We wish to define the line integral

$$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy.$$

This can be done in several ways. One of which is to partition the curve γ by a very large number of points, define Riemann sums for the partition, and then take a limit as the "mesh" of the partition gets finer and finer. However an argument like the one we did in class for complex line integrals shows that if $c(t) = (x(t), y(t))$ with $a \le t \le b$ is a smooth parameterization of γ then

$$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \big(P(x(t),y(t))x'(t) + Q(x(t),y(t))\big)\,dt.$$

This reduces computing $\int_\gamma P(x,y)\,dx + Q(x,y)\,dy$ to first finding a parameterization of γ and then computing a standard Riemann integral such as we have all seen in calculus.

Here is an example. Let γ be the upper half of the circle $|z| = 1$ transversed counter clockwise. Then compute

$$\int_\gamma xy^2\,dx - (x + y + x^3 y)\,dy. \tag{1}$$

This half circle is parameterized by

$$x(t) = \cos(t), \quad y(t) = \sin(t)$$

with $0 \le t \le \pi$. Now the calculation proceeds just as if we were doing a change of variables. That is

$$dx = -\sin(t)\,dt, \quad dy = \cos(t)\,dt$$

Using these formulas in (1) gives

$$\begin{aligned}
&\int_\gamma xy^2\,dx - (x+y+x^3y)\,dy \\
&\quad = \int_0^\pi \big(\cos(t)\sin^2(t)(-\sin(t))\,dt - (\cos(t)+\sin(t)+\cos^3(t)\sin(t))\cos(t)\,dt\big) \\
&\quad = \int_0^\pi \big(\cos(t)\sin^2(t)(-\sin(t)) - (\cos(t)+\sin(t)+\cos^3(t)\sin(t))\cos(t)\big)\,dt
\end{aligned}$$

With a little work[1] this can be shown to have the value $-(2/5+\pi/2)$.

Here is another example.

$$\int_C (2x+y)\,dx + (2y-x)\,dy$$

where C is the part of the curve $y = x^2+1$ between the points $(0,1)$ and $(2,5)$. This curve is parameterized by

$$x(t) = t \qquad y(t) = t^2+1.$$

Thus

$$dx = dt, \qquad dy = 2t\,dt$$

with $0 \le t \le 2$. Thus

$$\begin{aligned}
\int_C (2x+y)\,dx + (2y-x)\,dy &= \int_0^2 \big((2t+t^2+1)(1) + (2(t^2+1)-t)(2t)\big)\,dt \\
&= \int_0^2 (4t^3 - t^2 + 6t + 1)\,dy \\
&= \frac{14}{3}
\end{aligned}$$

Here are some for you to try.

1. $\displaystyle\int_\Gamma (x+2y)\,dx + 3x^2\,dy$ where Γ is the line segment form $4+3i$ to $7-i$.
2. $\displaystyle\int_\gamma (x+y)\,dx + y\,dy$ where γ is the lower right quarter of the circle $|z| = 3$ traversed clockwise.

Complex contour integrals. Let $f = u + iv$ be a complex valued function defined on a smooth curve γ. Then we have defined in class

[1] Here I am being a bit of a hypocrite as I used the computer package Maple to do the calculation

the contour integrals $\int_\gamma f(z)\,dz$. If $z = x + iy$ then $dz = dx + idy$ and we have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_\gamma (u+iv)(dx+i\,dy)\\ &= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dy + u\,dx.\end{aligned}$$

So computing a complex contour just reduces to computing two real line integrals. Thus all the remarks above apply to this case also.

Here is an example. Compute $\int_C z^2\,dz$ where C is the part of the curve $x = y^2$ between $(1,-1)$ and $(1,1)$. This curve is parameterized by

$$x(t) = t^2, \quad y(t) = t$$

with $-1 \le t \le 1$. For this problem it is smarted to write the parameterization in complex form:

$$z(t) = t^2 + it.$$

(Still with $-1 \le t \le 1$.) Then

$$dz = 2t\,dt + i\,dt = (2t+i)\,dt$$

Thus

$$z^2\,dz = (t^2+it)^2(2t+i)\,dt = \big((t^5-4t^3)+i(5t^4-t^2)\big)\,dt.$$

Therefore

$$\int_C z^2\,dz = \int_{-1}^{1} (t^5-4t^3)+i(5t^4-t^2)\,dt = \frac{4}{3}i$$

Here are more practice problems.

1. $\displaystyle\int_C |z|^2\,dz$ where C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.
2. $\displaystyle\int_\Gamma \frac{dz}{z}$ where Γ is the circle $|z| = r$ traversed in the counterclockwise direction.