Final Name:____________________________

Show your work! Answers that do not have a justification will receive no credit.

1. (20 points) Compute the following:
(a) All values of $\sqrt[3]{i}$.

(b) All values of i^{2i}.

(c) $\mathrm{Log}(-3-3i)$.

(d) All roots to $z^2 - 2iz - 10 = 0$.

2. (20 points) (a) Use $e^{(\alpha+\beta)i} = e^{\alpha i}e^{\beta i}$ to derive the addition formula for the sine function.

(b) Show that $|e^z| = e^{\mathrm{Re}\, z}$.

3. (20 points) (a) State the Cauchy-Riemannian equations.

(b) Show that if $f(z) = u + iv$ is analytic and $|f|^2 = u^2 + v^2$ is constant then f is constant.

4. (15 points) (a) Graph $|(1+i)z+2| = 4$

(b) What is the image of $D = \{z : 1 < |z| < 2, 0 < \text{Arg}(z) < \pi/4\}$ under the map $f(z) = z^4$. Graph both D and the image $f[D]$.

5. (20 points) Let D be a domain with smooth boundary and let $f(z)$ be analytic in D. Show that the Cauchy integral theorem $\int_{\partial D} f(z)\,dz = 0$ holds. You may use Green's theorem:
$\int_{\partial D}(P\,dx + Q\,dy) = \iint_D(-P_y + Q_x)\,dx\,dy.$

6. (20 points) (a) State the Cauchy integral formula.

(b) Evaluate $\displaystyle\int_{|z|=3} \frac{\cos(z)\, dz}{(z-1)(z^3-64)}$

(c) Evaluate $\displaystyle\int_{|z|=2} \frac{z\, dz}{(z-1)^2}$

7.(15 points) (a) What is the domain of analyticity of the function $f(z) = \dfrac{\sin(z^3)}{z(z^2-16)}$?

(b) For this function what is the radius of convergence if $f(z)$ is expanded as a power series about the point $z = 1 + 4i$.

8. (15 points)(a) Show that $u(x, y) = x^4 - 6x^2y^2 + y^4$ is harmonic.

(b) Find the harmonic conjugate to u.

9. (15 points) (a) Explain why $f(z) = \frac{1}{z}$ has an anti-derivative in $D = \{z : \operatorname{Re} z > 0\}$.

(b) Explain why $f(z) = \frac{1}{z}$ does not have an anti-derivative in $A = \{z : 1 < |z| < 3\}$.

10. (10 points) Find all solutions to $\cos(z) = \frac{5}{4}$.

11. (25 points) Let h be harmonic in the simply connected domain D.
(a) Show that $f(z) = h_x - ih_y$ is analytic in D.

(b) Explain why the function $f(z)$ has an anti-derivative in D.

(c) Show that h is the real part of an analytic function in D.

12. **Surprise mystery question: 10 free points** (and have a good winter break).