Mathematics 527 Final Name:____________________

You will need a calculator. You may use one page (both sides) of notes during the test.

1. (15 points) Find the Taylor series of $\sin(2x)$ about the point $x = \pi/4$.

2. (20 points)

(a) Find the Taylor expansion of $F(x) = \int_0^x e^{t^3}\,dt$ at $x = 0$.

(b) Find $\int_0^{1/3} e^{-t^3}\,dt$ to six decimal places and explain why your answer is correct.

3. (20 points) Let $f(x)$ be a function that has derivatives of all orders and let

$$\varphi(h) = \frac{f(x+h) - f(x-h)}{2h}.$$

(a) Use Taylor theorem to find an expression for the error in using $\varphi(h)$ for an approximation to $f'(x)$.

(b) Assume $\varphi(h)$ has an expansion

$$\varphi(h) = f'(x) + a_2(x)h^2 + a_4(x)h^4 + a_6(x)h^6 + \cdots$$

Then find an approximation to $f'(x)$ that is of order $O(h^4)$.

4. (15 points)

(a) Draw a picture and explain the geometry behind Newton's method.

(b) Give an example where Newton's method does not work. (A picture is acceptable.)

5. (15 points) Write a short essay comparing the good and bad points of Newton's method as compared to the bisection method.

6. (15 points) Let $f(x)$ be a continuous function with $f(-1) = 2$ and $f(3) = -5$. Then how many steps in the bisection method are required to get the answer correct to 6 decimal places?

7. (20 points) A function $f(x)$ is only known by the table of values:

x	1	2	5
$f(x)$	-1	1	3

(a) What is the polynomial $p(x)$ of degree ≤ 2 that interpolates this data.

(b) Give an estimate of $f(3)$ and give a brief explanation of why you believe this estimate is reasonable.

8. (10 points) Let $p_n(x)$ be the polynomial that interpolates the function $f(x) = e^{-x/2}$ on the interval $[0, 5]$ at $n+1$ equally spaced nodes. Then how large to we have to take n so that the error in approximating e^{-x} by $p_n(x)$ is $\leq 10^{-6}$?

9. (10 points) Explain geometrically what the trapezoid rules does to approximate an integral. (Give a picture.)

10. (10 points) Let $f(x, y)$ be a function so that

$$f(1,2) = 6, \quad f_x(1,2) = 5, \quad f_y(1,2) = 4,$$
$$f_{xx}(1,2) = 3, \quad f_{xy}(1,2) = 2, \quad f_{yy}(1,2) = 1.$$

Then give an approximation to $f(1.1, 1.8)$ using Taylor's theorem.

11. (20 points) Consider the initial value problem

$$\dot{x}(t) = -32 - \frac{t}{1 + x(t)^2}, \qquad x(0) = 100.$$

(a) Explain why there is a unique $t_* > 0$ so that $x(t_*) = 0$.

(b) Find an upper bound on t_*.

(c) Explain how you would go about computing t_* on a computer.