Work Sheet 6

Definition 1. A ***rate equation*** (also called a ***differential equation***) as an equation that involves a functions (say $y(t)$) and its derivative. (It might involve derivatives higher than the first.)

Some examples: The equation

$$y'(t) = t^2 + y^2$$

is a rate equation for the function $y(t)$. This could have been written as

$$\frac{dy}{dt} = t^2 + y^2.$$

The equation

$$\frac{dP}{dt} = .05P\left(1 - \frac{P}{400}\right)$$

is a rate equation for P. The equation

$$\frac{dA}{dr} = \frac{A^2 - r^2}{A^2 + r^2}$$

is a rate equation for A (where this time the independent variable is r).

A rate equation for $y(t)$ (or for some other variable, say $P = P(t)$) is ***time independent*** iff it is of the form

$$y' = f(y)$$

(that is the right hand side of the equation, that is $f(y)$, does not depend on the independent variable t.) Thus

$$\frac{dP}{dt} = P^2 + 3P^5 - e^P$$

is time independent as $P^2 + 3P^5 - e^P$ only depends on P and not on t, while

$$\frac{dP}{dt} = t + P$$

is not time time independent as $t + P$ does depend on t (so this equation is ***time dependent***).

If

$$y' = f(y)$$

is a time independent rate equation, then an ***equilibrium solution*** is a solution of the form $y = c$ there c is a constant. If $y = c$ is a solution, then $y' = c' = 0$ are therefore the rate equation $y' = f(y)$ becomes

$$0 = f(c).$$

Therefore the equilibrium solutions of $y' = f(y)$ are $y = c$ where c is a solution to $f(y) = 0$.

An equilibrium solution $y = c$ is ***stable*** iff $\lim_{t\to\infty} y(t) = c$ for all solutions $y(t)$ with $y(0)$ close to c (that is $y = c$ as a horizontal asymptote as t moves to the right). In all other cases $y = c$ is an ***unstable equilibrium***. See class notes for determining if a point is stable or unstable.

Problem 1. Find the equilibrium points of the following time independent rate equations and classify as to being stable or unstable. Probably the best way to do this is to graph the solutions by figuring out where they are increasing or decreasing.

(1) $y' = 3y$. **Answer:** $y = 0$ is the only equilibrium point and it is unstable.

(2) $y' = -.2y$. **Answer:** $y = 0$ is the only equilibrium point and it is stable.

(3) $\dfrac{dP}{dt} = .01P\left(1 - \frac{P}{100}\right)$. **Answer:** There are two equilibrium points $P = 0$ and $P = 100$. $P = 0$ is unstable and $P = 100$ is stable.

(4) $\dfrac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ where r and K are positive constants. Hint: Just think about what you did in the last problem with and replace .01 by r and 100 by K.

(5) $y' = y(y-1)(y-3)$. **Answer:** The equilibrium points are $y = 0$, $y = 1$, and $y = 3$. The only one that is stable is $y = 1$.

(6) If $y' = y(y-1)(y-3)$ (as in the last problem) if $y(t)$ is the solution with $y(0) = .8$ then give an estimate of $y(100)$.

(7) $\dfrac{dP}{dt} = (P-1)(P-2)(P-3)$. **Answer:** The equilibrium points are $P = 1$, $P = 2$ and $P = 3$. The only stable one is $P = 2$.

(8) $\dfrac{dP}{dt} = (P-a)(P-b)(P-c)$ where a, b, c are constants with $a < b < c$. Hint: This is really just the same as the last problem; think of a as 1, b as 2, and c as 3.

(9) $\dfrac{dP}{dt} = -(P-1)(P-2)(P-3)$. **Answer:** The equilibrium points are $P = 1$, $P = 2$ and $P = 3$. The stable ones are $P = 1$ and $P = 3$.

(10) $\dfrac{dP}{dt} = -r(P-a)(P-b)(P-c)$ where a, b, c, r are all positive and $a < b < c$. Hint: This is very much like the last problem.

(11) $P' = .1P\left(1 - \frac{P}{100}\right) - .2$. **Answer:** To find the equilibrium points, solve $.1P\left(1 - \frac{P}{100}\right) - .2 = 0$. This gives $P = 27.6393202$ and $P = 72.3606797$. Only $P = 72.3606797$ is stable.

Problem 2. Page 48 of the text, problems 2.1 and 2.2.