Work Sheet 1

We have seen that the solution to the difference equation

$$N_{t+1} = (1+r)N_t$$

is

$$N_t = N_0(1+r)^t$$

where N_0 is the initial size of the population. We now wish to solve the slightly more complicated equation

$$N_{t+1} = (1+r)N_t + S \tag{1}$$

where S is a constant (that might be negative) called the ***stocking rate***. What we do is let

$$N_t = U_t + c \qquad \text{which is the same as} \qquad U_t = N_t - c$$

where c is a constant to be determined. We use this in equation (1) to get

$$U_{t+1} + c = (1+r)(U_t + c) + S.$$

Doing some algebra reduces this to

$$U_{t+1} = (1+r)U_t + (rc + S).$$

We now choose c so that

$$(rc + S) = 0,$$

that is

$$c = -\frac{S}{r}.$$

Then the equation for U_t becomes

$$U_{t+1} = (1+r)U_t$$

and we know how to solve this:

$$U_t = U_0(1+r)^t.$$

Using $U_t = N_t - c$ gives

$$N_t - c = (N_0 - c)(1+r)^t,$$

but $c = -\dfrac{S}{r}$ so this becomes

$$N_t = \left(N_0 + \frac{S}{r}\right)(1+r)^t - \frac{S}{r}.$$

We summarize:

Theorem 1. *If $r \neq 0$ then the solution to*

$$N_{t+1} = (1+r)N_t + S$$

is

$$N_t = \left(N_0 + \frac{S}{r}\right)(1+r)^t - \frac{S}{r}.$$

□

Now assume that $-1 < r < 0$. Then $0 < 1 + r < 1$. Thus as t becomes very large, the number $(1 + r)^t$ is almost zero. Thus when for very large t we have

$$N_t = \left(N_0 + \frac{S}{r}\right)(1 + r)^t - \frac{S}{r} \approx \left(N_0 + \frac{S}{r}\right)0 - \frac{S}{r} = -\frac{S}{r}.$$

Thus the size of the population eventually levels out at size $-\frac{S}{r}$. Again we summarize:

Theorem 2. *Let $-1 < r < 0$ and S be any constant. The for any initial value N_0 the solution to*

$$N_{t+1} = (1 + r)N_t + S$$

will eventually level out at the value

$$N_\infty = -\frac{S}{r}.$$

□

Problem 1*.* Assume that a lake is being over fished for trout to the extent that the annually rate of increase of the trout population is $r = -.15$(trout/year)/trout. At what rate should the lake be stocked so that the trout population stays constant at 10,000 trout?

Solution. If N_t is the number of trout in the lake in the t-th year and the lake is stocked with S trout/year, then we want to find S so that the trout population levels off at 10,000 trout. The trout population will satisfy

$$N_{t+1} = (1 + r)N_t + S$$

which in our case, where $r = -.15$, this is

$$N_{t+1} = .85N_t + S.$$

The population will level off at a value of

$$N_\infty = -\frac{S}{r} = \frac{S}{.15}.$$

We want this value to be 10,000, that is

$$\frac{S}{.15} = 10{,}000$$

which leads to the answer

$$S = (.15)(10{,}000) = 1{,}500 \text{ trout/year}.$$

□