Homework assigned Friday, October 8

We wish to be able to find the stable age distribution for a Leslie matrix, A, more directly than just computing $A^t\vec{n}(0)$ for large values of t and seeing when it stabilizes. (If you are not interested in the theory, you can skip down to **Summary**.) Let us start with a 3×3 matrix

$$A = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

Now Leslie noticed that if

$$\vec{n} = \begin{bmatrix} 1 \\ \frac{P_1}{\lambda} \\ \frac{P_1 P_2}{\lambda^2} \end{bmatrix}$$

(where λ is a number to be determined later) then

$$A\vec{n} = \begin{bmatrix} F_1 + \frac{F_2 P_1}{\lambda} + \frac{F_3 P_1 P_2}{\lambda^2} \\ P_1 \\ \frac{P_1 P_2}{\lambda} \end{bmatrix} \tag{1}$$

Then if we choose λ so that

$$\frac{F_1}{\lambda} + \frac{F_2 P_1}{\lambda^2} + \frac{F_3 P_1 P_2}{\lambda^3} = 1,$$

which (multiply by λ) implies

$$F_1 + \frac{F_2 P_1}{\lambda} + \frac{F_3 P_1 P_2}{\lambda^2} = \lambda$$

$$A\vec{n} = \begin{bmatrix} \lambda \\ P_1 \\ \frac{P_1 P_2}{\lambda} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \frac{P_1}{\lambda} \\ \frac{P_1 P_2}{\lambda^2} \end{bmatrix} = \lambda\vec{n}$$

From this it is not hard to see that if $\vec{n}$ gives the stable age distribution.

Summary: If λ is a the positive solution to

$$\frac{F_1}{\lambda} + \frac{F_2 P_1}{\lambda^2} + \frac{F_3 P_1 P_2}{\lambda^3} = 1 \tag{2}$$

(this is the ***Lotka-Euler equation***) then the stable age distribution is given by

$$\vec{n} = \begin{bmatrix} 1 \\ \frac{P_1}{\lambda} \\ \frac{P_1 P_2}{\lambda^2} \end{bmatrix} \tag{3}$$

Example. Let

$$A = \begin{bmatrix} 0 & 2 & 10 \\ .1 & 0 & 0 \\ 0 & .9 & 0 \end{bmatrix}$$

so that

$$F_1 = 0, \quad F_2 = 2, \quad F_3 = 10, \quad P_1 = .1, \qquad P_2 = .9.$$

Thus the Lotka-Euler equation is

$$\frac{F_1}{\lambda} + \frac{F_2 P_1}{\lambda^2} + \frac{F_3 P_1 P_2}{\lambda^3} = \frac{(2)(.1)}{\lambda^2} + \frac{(10)(.1)(.9)}{\lambda^3} = \frac{0.2}{\lambda^2} + \frac{0.9}{\lambda^3} = 1.$$

Solve this using your calculator to get

$$\lambda = 1.034429638$$

Then the vector for the stable age distribution is

$$\vec{n} = \begin{bmatrix} 1 \\ \frac{P_1}{\lambda} \\ \frac{P_1 P_2}{\lambda^2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{.1}{1.034429638} \\ \frac{(.1)(.9)}{(1.034429638)^2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0.09667163075 \\ 0.08410863772 \end{bmatrix}$$

So

$$\text{Percent in Stage 1} = 100\frac{1}{1 + 0.09667163075 + 0.08410863772}\% = 84.69\%$$

$$\text{Percent in Stage 1} = 100\frac{1}{1 + 0.09667163075 + 0.08410863772}\% = 8.19\%$$

$$\text{Percent in Stage 1} = 100\frac{1}{1 + 0.09667163075 + 0.08410863772}\% = 7.12\%$$

bf Problem 1. Use this method to find the stable age distribution for the Leslie matrix

$$A = \begin{bmatrix} 0 & 2 & 6 \\ .4 & 0 & 0 \\ 0 & .1 & 0 \end{bmatrix}$$

Problem 2. Use this method to find the stable age distribution for the Leslie matrix

$$A = \begin{bmatrix} 0 & 25 & 400 \\ .01 & 0 & 0 \\ 0 & .2 & 0 \end{bmatrix}$$