Work Sheet 7

We now wish to use what we have learned to work to model some problems about population growth. Our most basic model is that without any outside constraints that the rate of growth of a population is proportional to the size of the population. That is

$$\frac{dP}{dt} = rP$$

where the constant of proportionality, r, is the ***intrinsic growth rate*** of the population. In this case we know the solution is

$$P(t) = P(0)e^{rt}$$

is exponential growth (when $r > 0$) or exponential decay (when $r < 0$). The graphs of solutions look like those in Figures 1 and 2

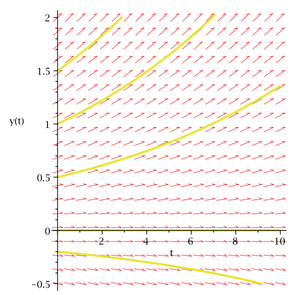

FIGURE 1. Exponential Growth: The solutions to $y' = .1y$ with the initial conditions $y(0) = -.2$, $y(0) = 0$, $y(0) = .5$, $y(0) = 1$, $y(0) = 1.5$

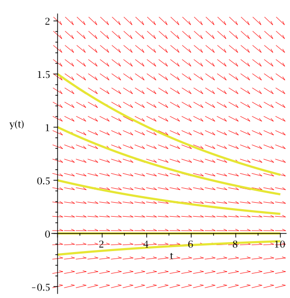

FIGURE 2. Exponential Decay: The solutions to $y' = -.1y$ with the initial conditions $y(0) = -.2$, $y(0) = 0$, $y(0) = .5$, $y(0) = 1$, $y(0) = 1.5$

Our other basic model is logistic growth. To review how we got this model we assume that the intrinsic growth rate for a small population was $r > 0$ and that the maximum size of the population that the environment would support was K (the ***carrying capacity***). We also supposed that the intrinsic growth rate depended on the size of the population. Let

$$R(P) = \text{intrinsic growth rate for a population of size } P.$$

Then the rate equation for the growth of the population is

$$\frac{dP}{dt} = R(P)P \tag{1}$$

We should have

$$R(0) = r \qquad \text{(As the intrinsic growth rate for small } P \text{ is } r\text{)}$$

Also

$$R(P) > 0 \quad \text{for } 0 < P < K \quad \begin{cases} \text{As the population size is less than} \\ \text{what the environment will support so} \\ \text{it is increasing} \end{cases}$$

$$R(P) < 0 \quad \text{for } K < P \quad \begin{cases} \text{As the population size is more than} \\ \text{what the environment will support, so} \\ \text{it is decreasing.} \end{cases}$$

So the graph of $R(P)$ as a function of P should look like any of the function in Figure 3

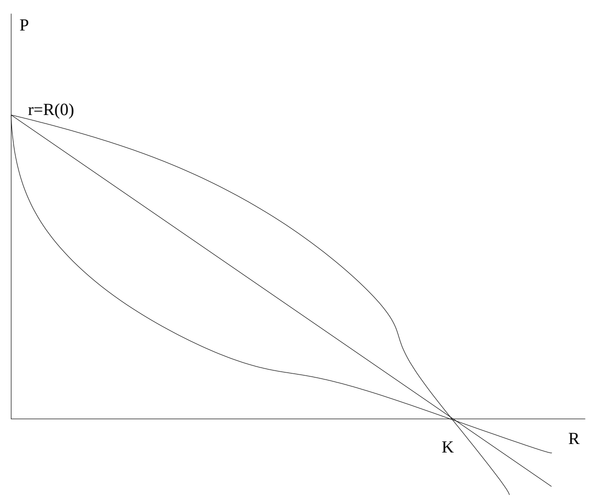

FIGURE 3

The simplest of these is the straight line. This has the equation

$$R(P) = r\left(1 - \frac{P}{K}\right).$$

Using this in equation (1) gives

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

which is the ***logistic equation***. This can be solved explicitly

$$P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-rt}} \tag{2}$$

but for much of what we will be doing having the explicit solution is less useful that just understanding how solutions behave.

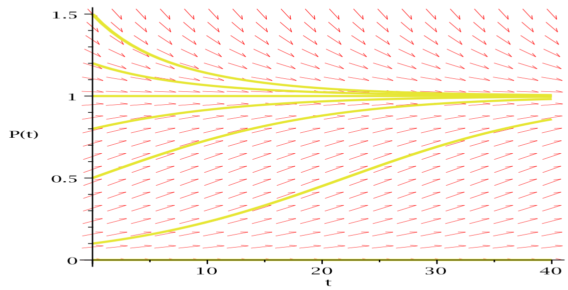

FIGURE 4. Logistic Growth: The solutions to $P'(t) = .1P(1 - P)$ with the initial conditions $P(0) = 0$, $P(0) = .1$, $P(0) = .5$, $P(0) = .8$, $P(0) = 1.0$, $P(0) = 1.2$, $P(0) = 1.5$. Note that in logistic growth all solutions with $P(0) > 0$ tend to the carrying capacity (in this case $K = 1$.)

Problem 1. Write down the logistic equation with intrinsic growth rate .03 and carrying capacity 500.

Solution. In this case $r = .03$ and $K = 500$ so the equation is

$$\frac{dP}{dt} = .03P\left(1 - \frac{P}{K}\right)$$

□

We now look at some models the extend these somewhat and let us make predictions how such things as harvesting and stocking effect populations.

Problem 2. A population of fish is a pond has an intrinsic growth rate of $-.03$ fish per fish per year and thus is dying out. If the pond stocked at the rate of 60 fish per year, find the stable equilibrium size of the population.

Solution. Let $P(t)$ be the size of the population after t years. We first write the new rate equation. It is

$$\frac{dP}{dt} = -.03P + 60.$$

We then find the equilibrium size of the population by setting $-.3P+60 = 0$ and solving for P. The solution is

$$P = \frac{-60}{-.03} = 2{,}000.$$

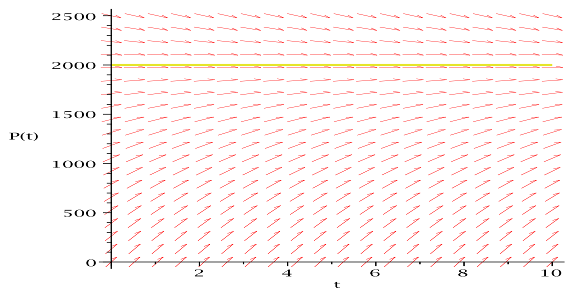

FIGURE 5. Graph showing where solutions to $\frac{dP}{dt} = -.03P + 60$ are increasing or decreasing.

Now draw the picture showing where the solution is increasing and decreasing. From this we see that the stable population is $P = 2{,}000$. □

Here are some related problems for you to try.

Problem 3. If the fish in our pound have an intrinsic growth rate of of $-.05$ fish per fish per year and the pond is stocked at a rate of 100 fish per year, find the stable equilibrium population of the pond. Do the same with intrinsic growth rate of of $-.12$ and a stocking rate of 300.

Problem 4. Hopefully this problem will convince you that algebra can make things easier. Assume that we have a population with an intrinsic growth rate of $r = -a$ where a is positive and we stock the population at a rate of S. Then what is the stable population size after we start stocking?

Outline of Solution. The rate equation is

$$\frac{dP}{dt} = -aP + S.$$

The equilibrium point is where $-aP + S = 0$. Solve for P to get

$$P = \frac{S}{a}$$

and then draw the picture to see that this is a stable equilibrium point. So the answer is $\frac{P}{a}$. □

We now look at a variant where we want to find the stocking rate.

Problem 5. Let lake have a population of large month bass that, due to fishing, has an intrinsic growth rate of $-.05$ per bass per year. The South Carolina Department of Natural Resources wishes to have stable population of 10,000 bass in the lake. At what rate should it be stocked?

Solution. $P(t)$ be the population size at time t and S be the stocking rate. Then the rate equation is

$$\frac{dP}{dt} = -.05P + S.$$

We find the equilibrium point by solving $-.05P + S = 0$ for S. This gives

$$P = \frac{S}{.05} = 20S.$$

(You should check that this is stable.) We want this to have the value 10,000. That is we want to solve $20S = 10{,}000$, which gives

$$S = \frac{10{,}000}{20} = 500$$

and this is the stocking rate we are looking for. □

Problem 6. Do the last problem which the intrinsic growth rate being $-.03$ bass per bass per year and where we want a stable population of 40,000. Try it with the intrinsic growth rate being $-.1$ per bass per year and where we want a stable population of 50,000.

Problem 7. This is another problem to show you that algebra is your friend. Assume that we have a population with an intrinsic growth rate of $r = -a$ with a positive and that we wish to have a stable population size of P_s. Then at what rate should we stock?

Proof. Let $P(t)$ be the population at time t and S the stocking rate. Then the rate equation is

$$\frac{dP}{dt} = -aP + S.$$

Solving $-aP + S = 0$ for P gives $P = \frac{S}{a}$ (and we should check that this is stable). We want this to have the value P_s. That is we want to solve

$$\frac{S}{a} = P_s$$

which has the solution

$$S = aP_s$$

which gives the required stocking rate. □

We now look at some models involving logistic growth.

Problem 8. Let a garden have a population of snails that has logistic growth with intrinsic growth rate of .1 snails per snail per year and a carrying capacity of 500. Toads are introduced to the garden and they eat 8% of the snail population per year. What happens to the size of the snail population?

Solution. Let $P(t)$ be the size of the snail population. Then the rate equation will be

$$\frac{dP}{dt} = .1P\left(1 - \frac{P}{500}\right) - .08P$$

To find the equilibrium points set $.1P\left(1 - \frac{P}{500}\right) - .08P = 0$. This factors to give

$$P\left(.1 - \frac{.1P}{500} - .08\right) = 0$$

So that $P = 0$ or $\left(.1 - \frac{.1P}{500} - .08\right) = 0$. Solving the latter gives

$$-\frac{.1P}{500} = .08 - .1 = -.02$$

so that

$$P = \frac{500}{.1}(.02) = 100.$$

Thus the equilibrium points are $P = 0$ and $P = 100$. You should check that $P = 0$ is unstable and $P = 250$ is stable. So the population of snails will be reduced to 100. $\square$

Problem 9. Do the last problem with the intrinsic growth rate of the snails being .05 snails per snail per year and a carrying capacity of 1,000 and the toads eating 4% of the the snail population per year.

Problem 10. The population of fish in a lake grows logistically with an intrinsic growth rate of .04 fish per fish per year and a carrying capacity of 10,000. At some point the fish are harvested are a rate of 50 fish per year. What happens to the fish population?

Solution. If $P(t)$ is the population at t years after the harvesting starts, then the rate equation is

$$\frac{dP}{dt} = .04P\left(1 - \frac{P}{10{,}000}\right) - 50$$

To find the equilibrium points we solve

$$.04P\left(1 - \frac{P}{10{,}000}\right) - 50 = 0.$$

This is a quadratic equation in P and the solutions are

$$P_1 = 1464.466094, \quad P_2 = 8535.533906$$

Thus the new stable population will be 8536 fish. So harvesting 50 fish a

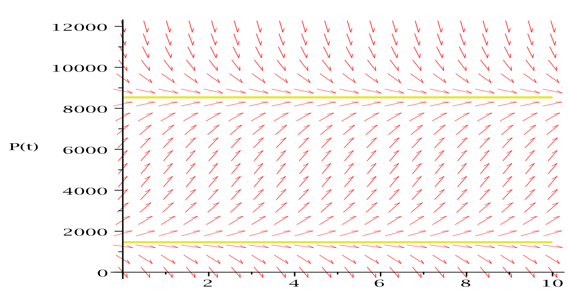

FIGURE 6. Graph showing that $P_1 = 1464.466094$ is unstable and $P_2 = 8535.533906$ is stable.

year reduces the population by 1,464 fish. Also note that if the population ever drops below $P_1 = 1464$ then the population will die off. $\square$

Problem 11. Do the problem above but change the harvesting rate to 200 fish per year.

Solution. This time the rate equation is

$$\frac{dP}{dt} = .04P\left(1 - \frac{P}{10{,}000}\right) - 200$$

and to find the equilibrium points we want to solve

$$.04P\left(1 - \frac{P}{10{,}000}\right) - 200$$

but this has no real numbers as solutions. (The solutions are the complex numbers $5{,}000 \pm 5{,}000i$). So there are no equilibrium points and $.04P\left(1 - \frac{P}{10{,}000}\right) - 200$ is always negative. Therefore the population will decrease down to zero. Thus the population will die off. $\square$

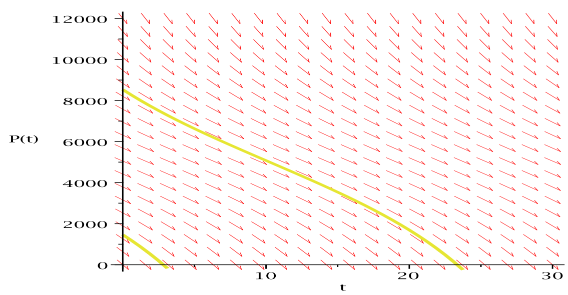

FIGURE 7. Graph showing that harvesting at the rate of 200 a year will kill off the population.

Problem 12. A populations of rabbits on an island grows logistically with an intrinsic growth rate of .05 rabbits per rabbits per year and a carrying capacity of 500. If the island is stocked at a rate of 20 rabbits per year, what happens to the population?

Solution. The rate equation is

$$\frac{dP}{dt} = .05P\left(1 - \frac{P}{500}\right) + 20.$$

We find the equilibrium by solving $.05P\left(1 - \frac{P}{500}\right) + 20 = 0$. This has solutions

$$P = -262.3475383, \quad P = 762.3475383$$

We only need to consider the positive solution. You can check that it is stable (see Figure 8). Therefore the population should tend to 762 rabbits. $\square$

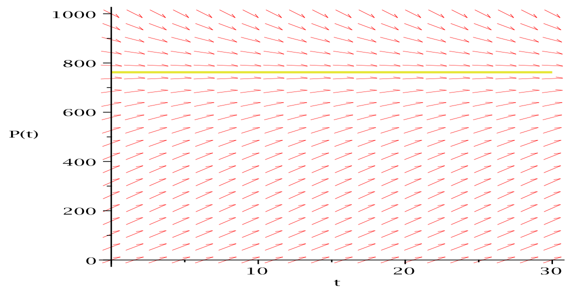

FIGURE 8. Graph showing that $P = 762.3475383$ is a stable equilibrium point.