Work Sheet 3

Let us generalize a result on the last worksheet. Assume that $y = y(t)$ is a function of t such that

$$y' = \rho(y - c)$$

where c is a constant. Then if we set $u = y - c$ and recall that the derivative of a constant is 0 we have that

$$u' = (y - c)' = y' = \rho(y - c) = \rho u.$$

But $u' = \rho u$ implies that

$$u = u(0)e^{\rho t}.$$

But $u = y - c$, and thus $u(0) = y(0) - c$ so the last displayed equation becomes

$$y - c = (y(0) - c)e^{\rho t}.$$

Adding c to both side gives

$$y = c + (y(0) - c)e^{\rho t}.$$

Problem 1. Find the solution to

$$y' = .3(y - 50).$$

Solution. In this case $\rho = .3$ and $c = 50$ and so

$$y = 50 + (y(0) - 50)e^{.3t}.$$

(Since $y(0)$ is not given we can not say any more about the solution.) □

Problem 2. Find the solution to

$$\frac{dP}{dt} = -.1(P - 100), \qquad P(0) = 500.$$

Solution. In this case we are using the variable P rather than y and $y' = \frac{dP}{dt}$, and $\rho = -.1$, $c = 100$ and $y(0) = P(0) = 500$. Thus

$$P = 100 + (500 - 100)e^{-.1t} = 100 + 400e^{-.1t}.$$

□

Problem 3. Find the solution to

$$\frac{dA}{dr} = .3(A - 200), \qquad A(2) = 400.$$

Solution. In this case $y = A$ and $t = r$, $c = 200$, and $\rho = .3$ and so

$$A = 200 + (A(0) - 200)e^{.3r}.$$

To find $A(0)$ we use that $A(2) = 400$ so that

$$400 = A(2) = 200 + (A(0) - 200)e^{.3(2)} = 200 + (A(0) - 200)e^{.6}.$$

Solving for $A(0)$ gives

$$A(0) = \frac{400 - 200}{e^{.6}} + 200 = 309.7623272$$

and thus

$$\begin{aligned} A &= 200 + (A(0) - 200)e^{.3r} \\ &= 200 + (309.7623272 - 200)e^{.3r} \\ &= 200 + 109.7623272e^{.3r} \end{aligned}$$

□

Problem 4. Solve

$$\frac{dN}{dt} = -.1N, \qquad N(0) = 300$$

Problem 5. Solve

$$\frac{dP}{dt} = -.1(P - 300), \qquad P(0) = 500.$$

Problem 6. Solve

$$y' = -.07(y - 30), \qquad y(4) = 50.$$

Use the solution to find $y(20)$. What is the approximate value of $y(t)$ when t is large (say $t > 100$)?

Problem 7. Suppose that

$$y' = -.1(y - c), \qquad y(0) = 500, \qquad y(5) = 400$$

find the value of c and a formula for y.

Proof. The formula for y is

$$y = c + (y(0) - c)e^{-.1t} = c + (500 - c)e^{-.1t}.$$

So

$$400 = y(5) = c + (500 - c)e^{-.5} = .3934693403c + 303.2653298$$

which we can solve for c to get

$$c = 245.8505919$$

and so

$$y = 245.8505919 + 254.1494081e^{-.1t}.$$

□

Problem 8. Suppose that

$$\frac{dP}{dt} = -.05(P - c), \qquad y(0) = 1{,}000, \qquad y(10) = 900$$

find the value of c and a formula for $P(t)$.