# Work Sheet 2

We have looked at exponential growth from a discrete point of view. That is looking at what each individual contributes to the population each year (or week, or day, depending on what the appropriate time period is. This makes sense for organisms that breed once pre year (or time period). Some examples would be

- Birds that nest in Northern regions such as the arctic. Most of these rase just one brood a year. This seems to be true for some larger birds nesting more Southernly such at hawks, owls and eagles. Some smaller local birds raise a couple of broods a year (which is the case for the robins in my backyard.)
- Cicadas (and numerous other insects). The annual (or dog day) cicada is the one that supplies the background noise to late summer here in Columbia. It has one brood a year. The 17 year cicada has one brood each 17 years (this is an example where the natural time period is not a year, but 17 years). Some butterflies only have one brood a year (monarch butterflies are an example).
- Deer, pigs, cows, and many other mammals just raise one brood a year. (Some large mammals, such as bears and elephants, take several years to raise their offspring and thus don't just breed once a year.)
- Annual plants in the temperate zones (where there is a cold winter) generally have one brood a year.
- Most trees in temperate zones just have one brood a year.
- Many fungi (a fact well known to mushroom hunters who know when to look for various types of mushrooms).

At the other extreme, there are organisms that are willing to breed year around. Examples are

- College students, and humans in general.
- In the right environment, mice, rats, lice, cockroaches, fleas, dust mites, and several other of humanity's habitual house guests.
- Most bacteria and protozoa.
- As in parts of the tropics there are no real seasons, and so many tropical species breed year around. An example of this are the fish of lakes Tanganyika and Malawi which have very stable environments.
- For the same reason, but I don't have data on this, I assume that many organisms in tropical reefs breed year round.
- Many fungi such as yeast and mildew.

We hare already seen that when a population grows at a per capita per year of $r$, that the population after $t$ years (which we assume is a whole number) is

$$P(t) = P_0(1+r)^t \qquad \text{where } P_0 \text{ is the current population size.}$$

If the growth rate is continuous let $P = P(t)$ be the population at time $t$, then the rate of change of the population is the derivative $\frac{dP}{dt}$. If we are assuming that the rate of growth is proportional to the size of the population, then

$$\frac{dP}{dt} = \rho P \tag{1}$$

for some constant $\rho$. We can solve this. Recall that

$$\frac{d}{dt}e^{ct} = \rho e^{ct}$$

for any constant $c$. So if (1) holds, then, by the product rule,

$$\frac{d}{dt}\left(e^{-\rho t}P\right) = -\rho e^{-\rho t}P + e^{-\rho t}\frac{dP}{dt} = -\rho e^{-\rho t}P + e^{-\rho t}\rho P = 0.$$

Thus the derivative of $e^{\rho t}P$ is zero. But if the derivative of a function is zero, then the function is constant. This implies that

$$e^{-\rho t}P = P_0$$

for some constant $P_0$. Multiply by $e^{\rho t}$ to get

$$P = P_0 e^{\rho t}. \tag{2}$$

Letting $t = 0$ shows that $P_0$ is the population size when $t = 0$. The number $\rho$ is called the ***i***ntrinsic growth rate. It is related to the ***p***er-capatia per year growth rate $r$ by

$$r = e^{\rho} - 1$$

The argument leading to equation (2) a more general result:

**Theorem 1.** *Let $r$ be a constant. If $y = y(t)$ is a function such that*

$$\frac{dy}{dt} = ry$$

*then*

$$y(t) = y(0)e^{rt}.$$

**Problem** 1. If the intrinsic rate of growth is $\rho = .1$, find the per-capita per year growth rate $r$. Do the same for the following values $\rho = .2$, $\rho = .01$, $\rho = -.1$, $\rho = -.03$.

**Problem** 2. If the per-capita per year growth is $r = .1$, find the intrinsic growth rate $\rho$. Do the same for the following values $r = .2$, $r = .01$, $r = -.1$, $r = -.03$.

**Problem** 3. A population grows without constraints. It starts with a population of 1,000 and five years latter has a population of 3,000. Find both the per-capita per year growth and the intrinsic growth rate and compute how long it takes for the population to reach 10,000.

**Problem** 4. Due to pollution a population of fish is dying off at an exponential rate. If the population starts at 500 and two years latter has size 400, then find both the Find both the per-capita per year growth and the intrinsic growth rate and compute how long it takes for the population to decline to 100.