Work Sheet 12

Here is a summary of some of the facts about the discrete logistic equation

$$N_{t+1} = N_t + RN_t\left(1 - \frac{N_t}{K}\right)$$

with per capita growth rate R and carrying capacity K. This is of the form

$$N_{t+1} = f(N_t)$$

where

$$f(x) = x + Rx\left(1 - \frac{x}{K}\right).$$

To get the equilibrium points solve $f(x) = x$ (or if you prefer the variable N, $f(N) = N$. There are two equilibrium points

$$N_* = 0, \qquad N_* = K.$$

To see if they are stable or unstable compute $f'(x)$ using the product rule

$$f'(x) = 1 + R\left(1 - \frac{x}{K}\right) + Rx\left(-\frac{1}{K}\right)$$

then

$$f'(0) = 1 + R > 1$$

so $N_* = 0$ is always unstable. And

$$f'(K) = 1 - R$$

So for $N_* = K$ to be stable we need $|1 - R| < 1$. As $R > 0$ this is equivalent to $R < 1$. We can be more precise.

- If $0 < R < 2$, then $N_* = K$ is a stable equilibirum point for any intial $N_0 \neq 0$ the polulation size for large t is $N_t \approx K$.
- If $2 < R < 2.45$ then $N_* = K$ is unstable but there is a stable ***two cycle***. That is the population will osculate between the two values

$$\left(\frac{2 + R + \sqrt{R^2 - 4}}{2R}\right)K, \qquad \left(\frac{2 + R - \sqrt{R^2 - 4}}{2R}\right)K.$$

- If $2.45 < R < 2.57$ there will be a stable cycle of order 4, 8, 16, 32 (in general of order 2^n for some n) depending on the size of R.
- When $2.57 < R$ then there are no stable cycles of any order and the system is chaotic.

Problems

(1) Draw a graph of a system that has exactly one equilibrium points and it is stable.

(2) Draw a graph of a system that has exactly one equilibrium points and it is unstable.

(3) If a system has exactly two equilibrium points is it possible for them to both be unstable?

(4) If a system has exactly two equilibrium points is it possible for them to both be stable? HINT: When the graph crosses the $y = x$ exactly twice. It will have to either go from above, to below and back to above; or from below to above and back to below. What does that say about stability?

(5) If a system has exactly three equlilbrium points, then is possable for all three to be stable? Can they all be unstable?

(6) For the system

$$N_{t+1} = 200 N_t e^{-.01t}$$

find the equilibrium points and decide if they are stable or unstable. (The hard part of this will be to figure out the correct window (i.e. choice of xmin and xmax) to use.)

(7) For the system

$$N_{t+1} = N_t + 1.5 N_t \left(1 - \frac{N_t + 20}{1{,}000}\right)$$

(8) What is the equation for a discrete logistic population with per capita growth rate of R, a carrying capacity of K that is being harvested at a rate of S organisms per year?

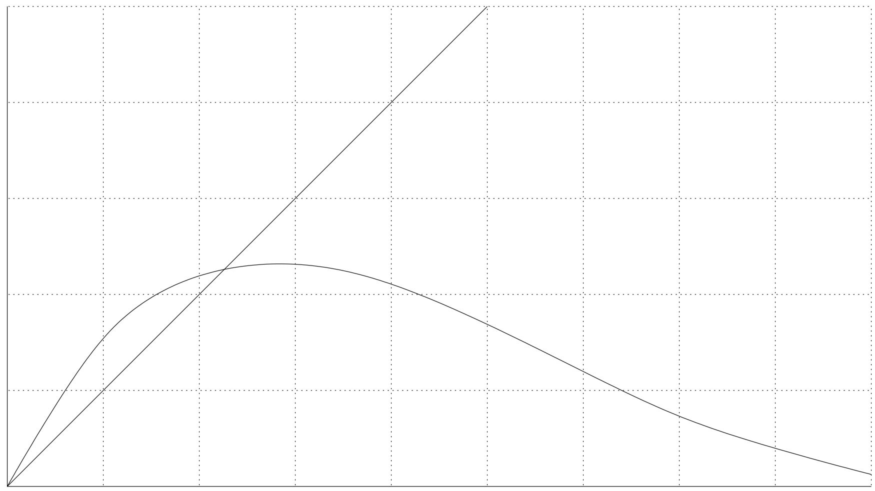

Figure 1. What happens to the population for large t?

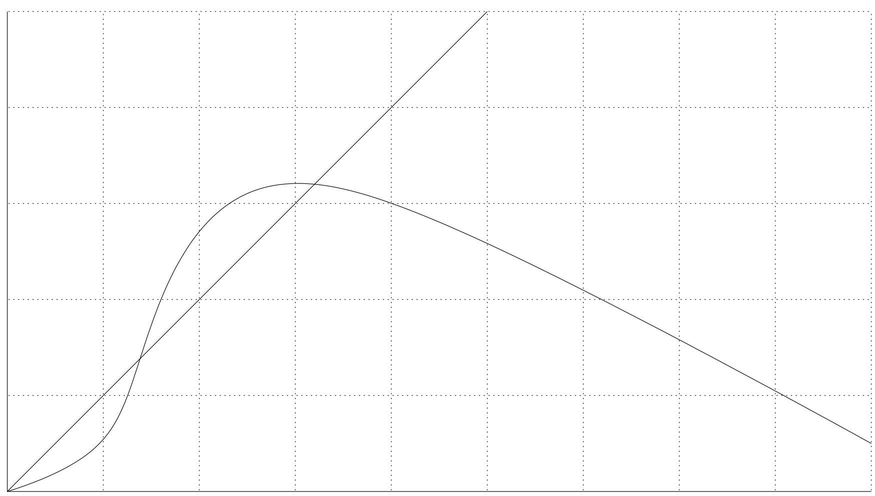

Figure 2. What happens to the population for large t?

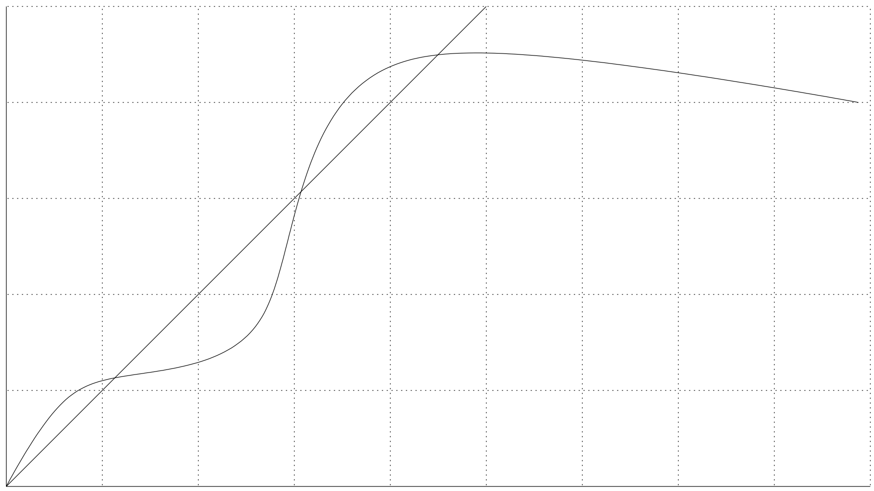

FIGURE 3. What happens to the population for large t?

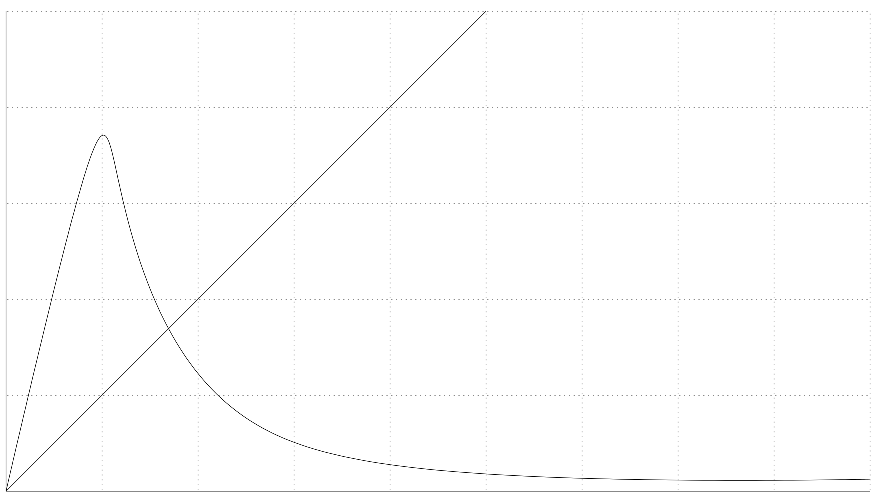

FIGURE 4. What happens to the population for large t?