Order of Magnitude Poker

In working with improper integrals we have had "plug $x = \infty$" into various functions. This really means that we are taking a limit. That is

$$\frac{3}{\sqrt{\infty}} = 0 \quad \text{because} \quad \lim_{x\to\infty} \frac{3}{\sqrt{x}} = 0$$

Here we give some rules that will help in doing this computations. Give two function $f(x)$ and $g(x)$ we say that $f(x)$ has a ***greater order of magnitude*** than $g(x)$ if and only if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \qquad \left(\text{this is the same as} \qquad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0\right).$$

Then $f(x)$ having greater order of magnitude than $g(x)$ means that for large values of x that the function $f(x)$ is much larger than the function $g(x)$. We say that $f(x)$ and $g(x)$ have the ***same order of magnitude*** iff there is a constant $L \neq 0$ so that

$$\lim_{x\to\infty} f(x)g(x) = L$$

We can now play a game—order of magnitude poker—as follows: I choose a function $f(x)$ and you choose a function $g(x)$. Then the one of us that has the function of with the greatest order of magnitude wins. If they have the same order of magnitude it is a tie. Then as a first idea of what beats what we have

exponentials e^{cx} with $c > 0$	beat	polynomials
	beat	logarithms
	beat	constants
	beat	reciprocals of logarithms
	beat	reciprocals of polynomials
	beat	exponentials e^{cx} with $c < 0$

Thus for example if I have $e^{x/1000}$ and you have $500x^{10000}$ then I win as

$$\lim_{x\to\infty} \frac{e^{x/1000}}{500x^{10000}} = \infty.$$

Using this bunch of ideas find the following

Problem 1.

$$\lim_{x\to\infty} x^{30}e^{-x/2} \qquad \lim_{t\to\infty} \frac{t}{\ln(t)} \qquad \lim_{x\to\infty} \frac{5000\ln(x)}{x+9}$$

$$\lim_{x\to\infty} \frac{2^x+1}{3^x} \qquad \lim_{x\to\infty} \frac{x^3-9x^2+1}{x^2+17} \qquad \lim_{x\to\infty} \frac{x}{\sqrt{x}}$$

(Some of these use more than just the rules above.)

This does tell the whole story. For example if we both get polynomials then the person with the polynomial of greatest degree wins. If the functions have the same order of magnitude then we have a tie. For example if I have $500x^5 + 2000$ and you have $x^6 - 2$ then you win as

$$\lim_{x\to\infty} \frac{x^6 - 2}{500x^5 + 2000} = \infty.$$

If two polynomials have the same degree then they have the same order of magnitude. For example $f(x) = 2x^7 - 9x^2 + 3$ and $g(x) = -4x^7 + 2x + 2$ have the same order of magnitude as

$$\lim_{x\to\infty} \frac{2x^7 - 9x^2 + 3}{-4x^7 + 2x + 2} = \frac{2}{-4} = -\frac{1}{2}.$$

Exercise 1. A ***rational function*** is the ratio of two polynomials. That is something that looks like $\dfrac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ are polynomials. (Examples $\dfrac{2x^3 + x - 9}{x^2 - 9}$, $\dfrac{5}{3x^7 - 2}$, x^4+2. (The last of these can be thought of as $\dfrac{x^4 + 2}{1}$.)) Give a complete set of rules for who wins when playing order of magnitude poker with rational functions. Also give some examples.

Exercise 2. Give the complete set of rules of who wins in order of magnitude poker when playing with functions that are sums of constants times exponentials. That is functions that look like $f(x) = c_1 e^{a_1} + \cdots + c_n e^{a_n x}$. (An example of such a function is $f(x) = -7e^{3x} + 4e^{x} + 5e^{-2x}$.)

Here are some practice problems

Problem 2.

$$\lim_{t\to\infty} \frac{5t^3 - 2t^2 + t - 5}{7t^3 + 17} \qquad \lim_{x\to\infty} \frac{e^{2x} + x^7}{4e^{2x} + e^{x}} \qquad \lim_{x\to\infty} \frac{\sqrt{2x+1}}{\sqrt{4x+2}}$$

$$\lim_{x\to\infty} \frac{\ln(x^3)}{\ln(2x^5)} \qquad \lim_{p\to\infty} \frac{\sqrt[5]{4x^2+3}}{5\ln(x)} \qquad \lim_{x\to\infty} \frac{500e^{2000x}}{e^{x^2/1000}}$$