Worksheet

To date we have only computed the the first derivatives of functions. But there is no reason to stop there, and in fact many and maybe even most rate equations that come up in physicical problems involve second derivatives. We will shortly have examples of this.

Let $y = f(x)$. Then the first derivative (or just derivative for short) is $y' = f'(x)$ rate of change of $f(x)$ and is somthing we are experts in computing. But then we can ask for the rate of change of the $f'(x)$. This is written as $y'' = f''(x)$. Now of course there is the third derivative $y'''(x) = f'''(x)$ etc. Derivatives of order four or more are usually writen as $f^{(4)}(x)$, $f^{(5)}(x)$ etc. Higher derivatives are also writen as

$$\frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \text{etc.}$$

Here are some examples:

$$\begin{aligned} y &= 3x^4 - 2x^3 + x - 5 \\ y' &= 12x^3 - 6x^2 + 1 \\ y'' &= 36x^2 - 12x \\ y''' &= 72x - 12 \\ y^{(4)} &= 72 \\ y^{(5)} &= 0 \end{aligned}$$

$$\begin{aligned} u &= t^2 e^{3t} \\ \frac{du}{dt} &= 2te^{3t} + 3t^2 e^{3t} \quad \text{(Product rule)} \\ &= (3t^2 + 2t)e^{3t} \\ \frac{d^2u}{dt^2} &= (6t + 2)e^{3t} + 3(3t^2 + 2t)e^{3t} \quad \text{(Product rule again)} \\ &= (9t^2 + 12t + 2)e^{3t} \end{aligned}$$

Likewise we can take the higher derivatives of fucntions of two or more variables. The notation here is

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = f_{xx}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = f_{xy}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = f_{yy},$$

Thus a fucntion of two variables has three second partial derivatives. (How many second partial derivatives does a function of three variables have?).

Here is some practice. Compute the second derivatives of the following functions. If it is a function of more than one variable, then find all the second partial derivatives.

$\dfrac{t^2 - 1}{\sqrt{5 + t + 3t^2}}$ $\qquad ts + \sqrt{2t + st^2}$ $\qquad \dfrac{y^3 - 1}{y - 1}$

$y = \dfrac{x - \sqrt{x^2 - 1}}{2}$ $\qquad \sqrt[3]{\dfrac{2 - 3x}{3 - 2x}}$ $\qquad \dfrac{3}{(2x^2 + 5x)^{\frac{3}{2}}}$

$\dfrac{1 - x^2}{1 + x^2}$ $\qquad x^2\sqrt{x^2 - a^2}$ (with a constant) $\qquad (x^2 + y^2)\tan(xy^2 z^3)$

$\dfrac{r(2 - \cos(2\theta))}{r^2 + z^2}$ $\qquad \sec(u^2 + 3v)$ $\qquad 5^{\sqrt{2 - x + 2y}}$

$x^2 e^{2y + 3z}\cos(4w)$ $\qquad (s - 3t)\cot(t)$ $\qquad \csc(5\theta)$

1. Show $u = e^x \sin(x)$ is a solution to the differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. (This is problem 7 page 293 of the text.)

2. For what value of the constant c is $z = x^2 + cy^2$ a solution to $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$?

3. Show $z = \frac{1}{\sqrt{t}} \exp \frac{-x^2}{4t}$ is a solution to the differential equation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}$. (This is problem 5 on page 293 of the text).

4. For what values of the constant r is $y = e^{rt}$ a solution to the equaiton $y'' - 3y' + 2y = 0$?