

Worksheet on Geometric Series

1. This problems relates to the finite geometric series

$$S = a + ar + ar^2 + \dots + ar^n.$$

(There are $n + 1$ terms.) The *first term* is a , the number r is called the *ratio* (note to get from one term to the next term you multiply by the ratio) and ar^n is the *last term*. Thus series has the sum

$$S = \frac{a - ar^{n+1}}{1 - r} = \frac{\text{first term} - \text{next term}}{1 - \text{next term}}$$

By the *next term* we mean ar^{n+1} which would be the next term if the series was continued. Generally it is easier to use the second form (given in terms of words rather than symbols). Use this formula to find the following sums:

- (a) $1 + 2 + 4 + 8 + \dots + 64$. (HINT: Here we get from one term to the next by multiplying by 2 so the *ratio* = 2, clearly *first term* = 1 and the last term is 64 so *next term* = $2 \cdot 64 = 128$. Thus

$$1 + 2 + 4 + 8 + \dots + 64 = \frac{\text{first term} - \text{next term}}{1 - \text{next term}} = \frac{1 - 128}{1 - 2} = 127.$$

Note we did not have to find n .)

- (b) $3 + \frac{3}{4} + \frac{3}{4^2} + \dots + \frac{3}{4^{30}}$.
(c) $P_0 + P_0(1+r) + P_0(1+r)^2 + \dots + P_0(1+r)^n$.
(d) $x - x^3 + x^5 - x^7 + \dots + (-1)^n x^{2n+1}$.

2. This problem is about the infinite geometric series

$$S = a + ar + ar^2 + ar^3 + \dots = \sum_{k=0}^{\infty} ar^k.$$

This will only sum up (or to use our jargon *converge*) if $|r| < 1$ or what is the same thing r is strictly between -1 and 1 . In this case the sum is

$$S = \frac{a}{1 - r} = \frac{\text{first term}}{1 - \text{ratio}}.$$

Again the form in terms of words is easier to use than the form $a/(1 - r)$. Use this to find the sums of the following

- (a) $4 - \frac{4}{3} + \frac{4}{3^2} - \frac{4}{3^3} + \frac{4}{3^4} - \dots$.
(b) $4x + 4x^3 + 4x^5 + 4x^7 + \dots$.
(c) $(1 - x) + \frac{1+x}{1-x} + \frac{1+x}{(1-x)^2} + \frac{1+x}{(1-x)^3} + \dots$
(d) $p + pq + pq^2 + pq^3 + \dots$

3. By taking the derivative of a geometric series term by term it is possible to get formulas for the sums of many other series. For example we know

$$1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x}.$$

By taking the derivative of this we get that

$$1 + 2x + 3x^2 + 4x^3 + \dots = \frac{1}{(1-x)^2}$$

If we now multiply both sides of this by x we get yet another formula

$$x + 2x^2 + 3x^3 + 4x^4 + \dots = \frac{x}{(1-x)^2}$$

Use similar tricks to find the sums of the following:

- (a) $1 + 3x^2 + 5x^4 + 7x^6 + 9x^8 + \dots$
- (b) $r - 4r^2 + 9r^3 - 16r^4 + 25r^5 + \dots$
- (c) $pq + 2pq^2 + 3pq^3 + 4pq^4 + 5pq^5 + 6pq^6 + \dots$

4. Rather like the last problem we can integrate a geometric series term by term to get a new series.

- (a) Use that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$ and that $\int \frac{dx}{1+x} = \ln(1+x) + C$ to find a series for $\ln(x)$.
- (b) Use that $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots$ and that $\int \frac{dx}{1+x^2} = \arctan(x) + C$ to find a series for $\arctan(x)$.

5. (This problem is taken, with minor changes, from *Elements of Algebra* by Leonard Euler and first published in 1770.) One *Sessa*, an *Indian*, having first invented the game of chess, shewed it to his prince, who was so delighted with it that he promised him any reward he should ask; upon which Sessa requested that he might be allowed one grain of rice for the first square on the chess board, two for the second, and so on doubling continually, to 64, the whole number of squares. How, supposing a pint to contain 7680 grains of rice, and one pint of rice to be worth 25cents, it is required to compute the value of the total number of grains of rice.