MATH 142 A Brief Introduction to Vectors Spring, 1995

While many quantities can be adequately described by a single number (mass, temperature, barometric pressure, speed, time, GPA, population), others seem to require two, or even more numbers for a full description. Some of these are velocity (how fast, and in which direction), force (how strong, and in which direction), multi-vitamins (how much of each of A, B, C, D, E, etc.), TV audience share (what % is watching each channel), animal or forest management (the number in each age or height class), income taxes (the population in each tax bracket), and so on. Indeed models of a modern economy may involve thousands of numbers to give a complete description. We will begin with the simple case of just two numbers, but as we shall see many of the ideas carry over without difficulty to the more general situation. You will quickly notice that this handout is not complete–many important ideas are explored only in the context of problems for you to work on. Bring your questions about these to class. In addition there will be the usual sorts of exercises on which you will get to practice computational skills.

A **vector** is just an ordered pair of real numbers: $\mathbf{v} = (a, b)$. Since boldface is not practical in handwritten work, one often writes $\overrightarrow{v}$, or $\underset{\sim}{v}$ or $\underline{\underline{v}}$ instead. There are a few special vectors that have symbols of their own: the **zero vector** is $\mathbf{0} = (0, 0)$, $\hat{\boldsymbol{\imath}} = (1, 0)$, $\hat{\boldsymbol{\jmath}} = (0, 1)$. Of course, it looks like there is a danger of confusing vectors with the coordinates of points in the plane; but it turns out that the context always makes things clear (and sometimes we even do want to have both meanings at the same time!). Our first example of a vector actually comes from geometry: if P is the point (x, y), and Q is the point (x', y'), then the **displacement vector** $\overrightarrow{PQ}$ is the vector $(x' - x, y' - y)$. All we are doing here is recording the net change in position of a particle if moves (somehow, not necessarily in a straight line) from the point P to the point Q. Note that since we are only considering *net change* in position, we can forget about the points P and Q; the same displacement could actually take place anywhere in the plane. It is convenient to represent such a vector geometrically as an arrow from the point P (which is called the tail) to the point Q (which is called the tip). We can calculate the length of this arrow very easily by the Pythagorean theorem to be $\sqrt{(x' - x)^2 + (y' - y)^2}$. More generally the **magnitude** or **size** of a vector $\mathbf{v} = (a, b)$ is $|\mathbf{v}| = \sqrt{a^2 + b^2}$. It is easy to see that the zero vector is the only one that has magnitude 0. It is conventional to put little hats on those vectors such as $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ that have magnitude 1; such vectors are called **unit vectors**.

By themselves vectors are not terribly interesting. What makes them useful is that we can perform many of the usual arithmetic or algebraic operations with them. Here are the two most basic ones. Let us take $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$.

(1) **Vector addition**: $\mathbf{v} + \mathbf{w} = (a + c, b + d)$.
(2) **Scalar multiplication**: If r is a real number, $r\mathbf{v} = (ra, rb)$.
(3) **Vector subtraction**: We abbreviate $(-1)\mathbf{v}$ as $-\mathbf{v}$. Then $\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v})$.

In the following problems you will investigate these operations.

Basic properties. Pretty much all the usual things work. Show that $\mathbf{v} + \mathbf{0} = \mathbf{v}$, $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, and that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$. Show that $0\mathbf{v} = \mathbf{0}$, $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$, and $(rs)\mathbf{v} = r(s\mathbf{v})$, where s is another real number. If $\mathbf{u}$ is yet another vector, show that $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$. We can also *decompose* vectors: show that $\mathbf{v} = a\hat{\imath} + b\hat{\jmath}$. In fact in some texts (usually in engineering it seems) the notation (a, b) is never used; instead all vectors are written in their decomposed form $a\hat{\imath}+b\hat{\jmath}$. The individual pieces $a\hat{\imath}$ and $b\hat{\jmath}$ are called the **components** of $\mathbf{v}$.

Geometric interpretation. Here we are working with displacement vectors. Convince yourself that vector addition of two displacements $\mathbf{v}$ and $\mathbf{w}$ corresponds to the overall displacement from the tail of $\mathbf{v}$ to the tip of $\mathbf{w}$ when we place the tail of $\mathbf{w}$ at the tip of $\mathbf{v}$. What happens if we perform the displacements by placing the tail of $\mathbf{v}$ at the tip of $\mathbf{w}$? Suppose you put the tails of $\mathbf{v}$ and $\mathbf{w}$ at the same point. Show that the arrow for $\mathbf{v} + \mathbf{w}$ (also for $\mathbf{w} + \mathbf{v}$) runs from this point to the opposite corner of the parallelogram that has $\mathbf{v}$ on opposite sides and $\mathbf{w}$ on the other two sides. This is called the *parallelogram law* for vector addition. It turns out to be a useful way to describe how forces combine, but for displacements the idea of one vector following after the other is better. This is especially true for vector subtraction: show how to represent $\mathbf{w} - \mathbf{v}$ as an arrow by considering that this is the vector that you need to add to $\mathbf{v}$ in order to get $\mathbf{w}$. Describe the geometric meaning of $r\mathbf{v}$; what happens when r is negative? Finally, show what vector decomposition means geometrically.

Vectors representing forces. Imagine two young Girl Scouts pulling a little red wagon loaded with Girl Scout cookies. They are each pulling on the handle with a force of 10 units (if you think about it, we don't have near as good intuition about units of force as we do time and distance–I sure have no idea how much a newton is on a human scale, and while ft-lbs are of human scale they are bad terminology for understanding physics correctly). How much force is exerted and in what direction if they are pulling at a 30° angle from the center line, one on one side and one on the other? How would your answer be different if the angle was 90° or 0°? What if one is pulling at 10° and the other at 20°?

Parametric equations and position vectors. Up to now when we have thought about motion it has always been *rectilinear*, that is, the particle was constrained to move back and forth along a straight line. In real life most motion occurs in two (or even three) dimensions. In this case its motion is described by two separate functions of time: $x(t)$ and $y(t)$. The equations $x = x(t)$ and $y = y(t)$ are called the parametric equations for the path of the particle, and t is called the parameter. Any such pair of equations in which x and y are expressed in terms of the same variable, which need not be time, are called parametric equations. We can track a moving particle by its displacement from the origin at time t; this is the **position vector** $\mathbf{r}(t) = (x(t), y(t)) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. This is our first example of a *vector-valued function*; instead of putting in a number t and getting a value $f(t)$, we are putting in a number and getting out a vector. Fortunately as we shall see such functions are no harder to work with than ordinary functions.

Velocity vectors. Imagine a particle traveling along a curved path in the plane. Fix a time t and draw the vectors $\mathbf{r}(t)$, $\mathbf{r}(t + \Delta t)$, and $(1/\Delta t)(\mathbf{r}(t + \Delta t) - \mathbf{r}(t))$ for a moderate size Δt (of course you'll want to use what you learned about vector subtraction and scalar multiplication above). Do this for a smaller Δt, and then for an even smaller one. What seems to be happening geometrically as Δt approaches zero? If you want to make things more specific, try using $\mathbf{r}(t) = (t, t^2)$. What curve is this particle following? We shall actually define the **derivative of the position vector**, or by another name the **velocity vector**, to be $\mathbf{v}(t) = \mathbf{r}'(t) = \lim_{\Delta t \to 0}(1/\Delta t)(\mathbf{r}(t + \Delta t) - \mathbf{r}(t))$.

In the specific example, can you confirm that $\mathbf{v}(t) = \mathbf{r}'(t) = (1, 2t)$? It is not to hard to see that in general for $\mathbf{r}(t) = (x(t), y(t)) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$, we obtain $\mathbf{r}'(t) = (x'(t), y'(t)) = x'(t)\hat{\boldsymbol{\imath}} + y'(t)\hat{\boldsymbol{\jmath}}$. Physically what this is saying is that velocity can be computed in each component separately. In general, the derivative of any vector-valued function $\mathbf{f}(t) = (f_1(t), f_2(t))$ is simply computed by $\mathbf{f}'(t) = (f_1'(t), f_2'(t))$, that is, one computes the derivative in each component separately.

One of the most important facts about the derivative in the first term was the microscope equation. It holds for vector valued functions also:

$$\Delta\mathbf{f} \approx \mathbf{f}'(a)\Delta t$$

where $\Delta\mathbf{f} = \mathbf{f}(a + \Delta t) - \mathbf{f}(a)$ is the change in $\mathbf{f}$ and Δ is the change in t.

Dot product. So far we have avoided multiplying vectors, and there is a very good reason for this! There just isn't any way to do it that behaves like ordinary multiplication. There are, however, certain operations that behave a little bit like multiplication. The first of these is called the **dot product** or **scalar product**. For $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$ the dot product is $\mathbf{v} \cdot \mathbf{w} = ac + bd$. The first thing to notice is that the answer is no longer a vector–*the dot product of two vectors is a number, that is, a scalar!* Another peculiarity is that the dot product can be zero, even when neither vector is zero; consider, for instance, $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}$. Nevertheless there are several properties that continue to be true for this funny product. If r is a real number, $\mathbf{v}$ and $\mathbf{w}$ are as above, and $\mathbf{u} = (e, f)$, show that $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$, $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, and $(r\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (r\mathbf{w}) = r(\mathbf{v} \cdot \mathbf{w})$. Perhaps the most interesting result is that the dot product of a vector with itself is related closely to its length: show that $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

Projection and orthogonal decomposition. If you look back at the Girl Scout problem, perhaps you can see that the pulling force that really counted was the part of it that was directed forward; the part that pulled sideways was wasted, so to speak. What do we mean by talking about these "parts"? The basic idea goes back to the decomposition of a vector that we mentioned earlier, only instead of using $\hat{\boldsymbol{\imath}}$

and $\hat{\boldsymbol{\jmath}}$ we will use perpendicular vectors that are more appropriate to our situation. Suppose one of those Girl Scouts is sitting on a nice smooth slide (if you must think like a physicist, set a heavy block on a frictionless inclined plane). If her mass is 20 Kg, then the force of gravity $\mathbf{F}$ pulling her directly vertically has a magnitude of 196 newtons (i.e., 196 Kg-m/sec^2; this comes from the equation $\mathbf{F} = m\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity, directed vertically with a magnitude of 9.8 m/sec^2). We can view this force as a vector sum of two separate forces, one parallel to the slide which causes her to move, and the other perpendicular to the slide, pulling her right into the slide (of course this is cancelled out by the resistance of the slide itself, which has to do with the molecular structure of the metal and so on). Let us call the parallel component $\mathbf{F}_{||}$ and the perpendicular component $\mathbf{F}_{\perp}$. We are simply saying that there is a decomposition $\mathbf{F} = \mathbf{F}_{||} + \mathbf{F}_{\perp}$. Such a decomposition is called *orthogonal* because the components are perpendicular (orthogonal, normal) to one another.

It would be instructive at this point to make yourself some more diagrams that illustrate these vectors for slides at different inclinations θ. Our problem is to find a way to compute the magnitude of the effective force $\mathbf{F}_{||}$. First, show that the angle between $\mathbf{F}$ and $\mathbf{F}_{\perp}$ is just θ. Then it is clear that $|\mathbf{F}_{\perp}| = |\mathbf{F}|\cos\theta$ (why?). Finally, how can you find $|\mathbf{F}_{||}|$? Now that you know the force on the girl, you can compute her acceleration, velocity, position, and so on (but since this isn't a physics class we won't do all that). By the way, this is how Galileo actually did his experiments on the effects of gravity; he didn't just drop balls off towers!

The story doesn't end here. Suppose we didn't know the angle θ, but we were in a situation where vectors $\mathbf{F}$ and $\mathbf{D}$ were known, and we wanted to decompose $\mathbf{F}$ into components parallel and perpendicular to $\mathbf{D}$. This occurs, for example, if we want to calculate the work done by the force $\mathbf{F}$ in displacing an object by an amount $\mathbf{D}$. Since only the force in the direction of $\mathbf{D}$ contributes to the work, $W = \text{(force)(distance)} = |\mathbf{F}_{||}||\mathbf{D}| = (|\mathbf{F}|\cos\theta)|\mathbf{D}|$, and we need to compute either $\mathbf{F}_{||}$ or $\cos\theta$. (You might wonder why force in a certain direction doesn't cause motion in exactly the same direction–but think about pushing a shopping cart whose wheels are locked in the wrong direction, or the force of gravity on a person constrained to move along a slide!)

According to the Law of Cosines, the general property of vectors that $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$, and the various other algebraic properties of the dot product mentioned above, we have

$$\begin{aligned} |\mathbf{F}-\mathbf{D}|^2 &= |\mathbf{F}|^2 + |\mathbf{D}|^2 - 2|\mathbf{F}||\mathbf{D}|\cos\theta \\ (\mathbf{F}-\mathbf{D})\cdot(\mathbf{F}-\mathbf{D}) &= \mathbf{F}\cdot\mathbf{F} + \mathbf{D}\cdot\mathbf{D} - 2|\mathbf{F}||\mathbf{D}|\cos\theta \\ \mathbf{F}\cdot\mathbf{F} - \mathbf{F}\cdot\mathbf{D} - \mathbf{D}\cdot\mathbf{F} + \mathbf{D}\cdot\mathbf{D} &= \mathbf{F}\cdot\mathbf{F} + \mathbf{D}\cdot\mathbf{D} - 2|\mathbf{F}||\mathbf{D}|\cos\theta \\ -2\mathbf{F}\cdot\mathbf{D} &= -2|\mathbf{F}||\mathbf{D}|\cos\theta, \end{aligned}$$

which leads to the formula that shows why the dot product is so important:

$$(\bigstar) \qquad \boxed{\mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta.}$$

If we know $\mathbf{F}$ and $\mathbf{D}$ the left hand side is very easy to compute–it's just a little easy arithmetic. The right hand side contains the geometric (length and angle) information, and in particular from it we can compute the cosine of the angle between the two vectors. (As you might imagine, one frequently can exploit this by making "two hands" arguments.) Then we see that the formula for work becomes very simple: $W = |\mathbf{F}||\mathbf{D}|\cos\theta = \mathbf{F}\cdot\mathbf{D}$. But now we should get a little bit suspicious. Dot products can turn out to be negative. What would negative work mean? The answer turns out to be fairly simple. In all the pictures so far θ has been an acute angle ($0 \le \theta \le \pi/2$), and therefore $\cos\theta$ has not been negative. But we might have a situation like this:

Then $\cos\theta < 0$, and the work comes out to be negative. So our very first formula for work wasn't quite correct: we should have said $W = \pm|\mathbf{F}_{\|}||\mathbf{D}|$, where the sign is $+$ if $\mathbf{F}_{\|}$ is aligned with $\mathbf{D}$, and $-$ if it is backwards. Roughly speaking, the idea is this: if an apple falls to the ground, gravity has done some work (making the apple fall faster and faster, or to be fancy, giving the apple a nice dose of kinetic energy which you can feel if it bonks you). But if we pick the apple up, *we* are doing the work (against gravity, or again to be fancy, we are giving the apple potential energy); in this sense our moving the apple opposite to the force of gravity is undoing the work that gravity did. [If this all sounds ridiculous, ask your physics professor for a better explanation.]

Anyhow, forgetting about work, if $\mathbf{F}$ and $\mathbf{D}$ are just a couple of vectors, we do get a nice formula for $\mathbf{F}_{\|}$, the **vector projection** of $\mathbf{F}$ on $\mathbf{D}$. If $\hat{\mathbf{u}}$ is a unit vector in the direction of $\mathbf{D}$, then $\mathbf{F}_{\|} = \pm|\mathbf{F}_{\|}|\hat{\mathbf{u}} = (|\mathbf{F}|\cos\theta)\hat{\mathbf{u}}$. As we saw in the exercises, $\hat{\mathbf{u}} = (1/|\mathbf{D}|)\mathbf{D}$, so

$$\begin{aligned} \mathbf{F}_{\|} = \left(\frac{|\mathbf{F}|\cos\theta}{|\mathbf{D}|}\right)\mathbf{D} &= \left(\frac{|\mathbf{F}||\mathbf{D}|\cos\theta}{|\mathbf{D}|^2}\right)\mathbf{D} \\ &= \left(\frac{\mathbf{F}\cdot\mathbf{D}}{\mathbf{D}\cdot\mathbf{D}}\right)\mathbf{D}. \end{aligned}$$

Later on we shall find many applications for vector projection. One should also note that the signed length of $\mathbf{F}_{||}$, or $|\mathbf{F}|\cos\theta$, is sometimes called the **scalar projection** of $\mathbf{F}$ on $\mathbf{D}$. If someone asks you to compute "the projection" you should always check which of these quantities they mean. Just one last remark: once we have computed $\mathbf{F}_{||}$ we can immediately compute $\mathbf{F}_{\perp}$ (how?), and hence obtain the orthogonal decomposition of $\mathbf{F}$ relative to $\mathbf{D}$. (Whew! that was a long section–don't panic if it seems a little much at first.)

Transformations. So far we have seen functions that accept a number t as input and produce a vector $\mathbf{f}(t)$ as output. We can go go one step further by considering functions that take vectors as input and produce vectors as output. If we think of vectors as displacements from the origin, and just look at what happens to their tips, such a function amounts to putting in points of the plane and getting out points of the plane. In other words, we have a certain transformation of the plane. This isn't really so strange. For example, one transformation that you are all familiar with is rotation, say by $\pi/2$ in the counterclockwise direction. This transformation carries $\mathbf{0}$ to $\mathbf{0}$, $\hat{\boldsymbol{\imath}}$ to $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{\jmath}}$ to $-\hat{\boldsymbol{\imath}}$. How can we describe such transformations? One way is to give a formula. The rotation transformation can be described by $T(x, y) = (-y, x)$ (check that this works!). Can you produce a formula for reflection across the x-axis? How about a rotation by $\pi/6$ counterclockwise? Or one that pushes every point twice as far as it was from the origin, but along the same line through the origin? We will only deal with *linear transformations* for now; these are ones that have formulas of the type $T(x, y) = (ax + by, cx + dy)$, and include all the examples we have mentioned so far.

It turns out there is an efficient way to represent these transformations. We simply collect all the coefficients in a tidy little box called a **matrix**, or an **array**, if you happen to be a computer person. In this case we have $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We also rewrite the vector $\mathbf{v} = (x, y)$ as the matrix $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ (the first form is known as a row vector and the second as a column vector–it is useful to switch back and forth). Then $T(x, y) = T(\mathbf{v})$ is given by matrix multiplication $A\mathbf{v} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$. Notice the result is a column vector, but if we want to we can switch it back to row vector form $(ax + by, cx + dy)$. (By the way, Maple is really adamant about this; it hates writing column vectors if it can possibly avoid it, maybe because it takes up too much space on the screen. It will insist that $A\mathbf{v}$ is $(ax+by, cx+dy)$.) To remember how matrix multiplication works, it is handy to think of it as really nothing more than a bunch of dot products. Each row (vector) inside A gets "dotted" in its turn with the column vector $\mathbf{v}$. You should check that our first rotation example comes from using the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Flipping across the x-axis comes from the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In the exercises you will get to work out some more examples.

So far the transformations that we have considered are so nice that we can understand how they work on the whole plane (which, by the way mathematicians like to abbreviate by $\mathbb{R}^2$) all at once. But some more complicated transformations are easier to understand if we look at how they transform just a piece of the plane such as the unit square with corners at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.