Algebra Propositions/Theorems, Exam 1

Packet One (1-38) Let G be a group, $a, b \in G$

• The inverse of $x \in g$ is unique.

• If $xy = 1$ or $yx = 1$, then $y = x^{-1}$.

• We can solve $ax = b$ and $xa = b$.

• The inverse of $(x_1 x_2 ... x_n)$ is $(x_n^{-1} x_{n-1}^{-1} ... x_1^{-1})$

• Right and left cancellation laws.

• $H \subseteq G$ is a subgroup $\iff$ $H \neq \emptyset$ and $a, b \in H \rightarrow ab^{-1} \in H$.

• $H \subseteq G$ is a subgroup $\iff$ $H \neq \emptyset$ and H is closed under products, then H is a subgroup of G.

• The left coset, xH is $xa : a \in H$

• If $H \leq G$, then any two cosets xH and yH of H have the same number of elements.

• **Lagrange's theorem/basic counting formula** Let $H \leq G$. Then, $|G| = [G : H]|H|$.

• If $|G| = p$, where p is a prime, then G is cyclic and $\forall a \in G, a \neq \emptyset, G =< a >$.

• If $H \leq G$, then the following are equivalent:
1 $xHx^{-1} = H, \forall x \in G$.
2 $xHx^{-1} \subseteq H, \forall x \in G$.
3 $xH = Hx \forall x \in G$.
4 $x \equiv y \ mod \ H \rightarrow xz = yz \ mod \ H$.
5 $x \equiv y \ mod \ H \rightarrow x^{-1} \equiv y^{-1} \ mod \ H$.

• Let $H \triangleleft G$. Then the set of cosets $\frac{G}{H}$ becomes a group with operation $(xH)(yH) = xyH$.

• If $H \leq G$, with $[H : G] = 2$, then H is normal in G.

• If $\phi : G \rightarrow H$ is a homomorphism, then ϕ maps the identity to the identity and inverses to inverses.

• Let $\phi : G \to H$ be a homomorphism. Then, $Im(\phi)$ is a subgroup of H and $Ker(\phi)$ is a normal subgroup of G.

• A homomorphism $\phi : G \to H$ is one to one or injective $\iff$ $Ker(\phi) =< 1 >$.

• If $\phi : G \to H$ is an isomorphism, so is the inverse $\phi^{-1} : H \to G$.

• **First Isomorphism Theorem** Let $\phi : G \to H$ be a homomorphism of groups, and let $K = Ker(\phi)$. Then $\frac{G}{K \cong Im(\phi)}$.

• **Correspondence Theorem** Let $\phi : G \to G'$ be an onto/sujective group homomorphism. Let $K = Ker(\phi)$. Then the below map gives a bijective corresondence between subgroups H' of G' and subgroups H of G that contain K. H' is normal $\iff$ $\phi^{-1}[H']$ is normal.

$$H' \longmapsto \phi^{-1}[H'] = x \in G : \phi(x) \in H'$$

• **Second Isomorphism Theorem** Let $H, K \leq G$ with K normal in G. Then HK is a subgroup of G, $K \triangleleft HK$, $H \cap K \triangleleft H$, and

$$\frac{HK}{K} \equiv \frac{H}{H \cap K}.$$

Packet 2 (39-45) Let G be a group.