Real Number Labelings for Paths and Cycles *

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Abstract

The problem of radio channel assignments with multiple levels of interference depending on distance can be modeled using graph theory. The authors previously introduced a model of labeling by real numbers. Given a graph G, possibly infinite, and real numbers $k_1, k_2, \ldots, k_p \geq 0$, a $L(k_1, k_2, \ldots, k_p)$-labeling of G assigns real numbers $f(x) \geq 0$ to the vertices x, such that the labels of vertices u and v differ by at least k_i if u and v are at distance i apart. We denote by $\lambda(G; k_1, k_2, \cdots, k_p)$ the infimum span over such labelings f. When $p = 2$ it is enough to determine $\lambda(G; k, 1)$ for reals $k \geq 0$, which will be a piecewise linear function. Here we present these functions when $p = 2$ for paths, cycles, and wheels.

1 Introduction.

Hale (1980) [10] used graph theory to model the efficient assignment of numerical channels to a network of transmitters such that interference between nearby transmitters is avoided. Transmitters can be represented by vertices, with vertices for nearby transmitters joined by an edge. The goal is to minimize the span of the assignment between the largest and smallest channels. The resulting graph theory problems concern "generalized colorings", in which the colors are integers. Now the difference between integer labels is a concern, whereas in traditional coloring we only care whether vertices receive the same or different colors.

In the late 1980's Lanfear described to Roberts [12] a channel assignment problem of this kind in which there are *two* levels of interference, depending on the distance

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between the transmitters. In the basic problem, channels for close locations must be distinct (differ by at least one), while those for very close locations must differ by at least two (due to spectral spreading). Griggs introduced the analogous "lambda-labeling" problem for graphs, and made the initial investigation of this graph theory problem with Yeh [9]. The generalization of the problem in which integer separations are specified at distances $1, 2, \ldots, p$ was also introduced, and a sizable literature has grown considering these labelings of graphs (see [5, 3, 11] for overviews).

Over the past several years considerable attention to such problems has arisen due to the growth of large networks for wireless and mobile communications. Often, the transmitter towers are laid out in large regular arrays that can be modeled effectively by the graph interpretation above. Since there is no reason why the channels and separations have to be restricted to integers, Griggs [7] proposed a more general real number model for labeling graphs with distance conditions, which he has been investigating jointly with Xiaohua Teresa Jin. The purpose of this article is to obtain the optimal spans of labelings with arbitrary conditions at distance two for two fundamental graph families, paths and cycles, and a closely related family, wheels. In the real number model, scaling allows us to reduce these problems for each graph to one with just a single parameter. These formulas will be valuable for the future development of the theory of labelings with distance conditions.

For the remainder of this section, we shall review the basic definitions of real number labelings, along with the tools from the theory that will be helpful. In Section 2 we shall collect all of our results, and compare them with what is already in the literature. The following three sections contain the proofs of the formulas for paths, cycles, and wheels, respectively. The proofs are necessarily complicated, as would be expected when one examines the surprisingly rich piecewise-linear functions for the optimal spans. For instance, the formula for the n-cycle depends on n modulo 12 for $n \geq 6$.

Griggs [9](1988) proposed studying the graph-theoretic analogue of the problem, which he extended in the natural way, by specifying separations $k_1, \ldots, k_p$ for vertices at distances $1, \ldots, p$: Specifically, we say a $L(k_1, k_2, \cdots, k_p)$**-labeling** of a graph G is an assignment of nonnegative numbers $f(v)$ to the vertices v of G, such that $|f(u) - f(v)| \geq k_i$ if u and v are at distance i in G. We say that labeling f belongs to the set $L(k_1, k_2, \cdots, k_p)(G)$. We denote by $\lambda(G; k_1, k_2, \cdots, k_p)$ the minimum span over such f, where the span is the difference between the largest and smallest labels $f(v)$. Griggs and Yeh [9] concentrated on the fundamental case of $L(2, 1)$-labelings, and many authors have subsequently contributed to the literature on these labelings (see [5, 7, 11]). Increasing attention has been paid recently to more general $L(k_1, k_2, \cdots, k_p)$-labelings.

Let us fix a graph G, which we assume to be finite in this paper, though the theory considers infinite graphs as well. Let us also specify the separations, which are real numbers $k_1, \ldots, k_p \geq 0$. We say a $L(k_1, k_2, \cdots, k_p)$**-labeling** of G is an assignment of real numbers $f(v)$ to the vertices v of G, such that $|f(u) - f(v)| \geq k_i$ if u and v are at distance i in G. We say that labeling f belongs to the set $L(k_1, k_2, \cdots, k_p)(G)$. We denote by $\lambda(G; k_1, k_2, \cdots, k_p)$ the infimum span over such f, where the span is the difference between the largest and smallest labels $f(v)$.

Clearly, such labelings exist for all $k_1, \ldots, k_p$ for a finite graph G, and so the infimum, $\lambda(G; k_1, k_2, \cdots, k_p)$, exists as well. Griggs and Jin proved the existence of an optimal labeling of a nice form, in which all labels belong to the discrete set, denoted by $D(k_1, k_2, \ldots, k_p)$, of linear combinations $\sum_i a_i k_i$, with nonnegative integer coefficients a_i:

Theorem 1.1 (The D-Set Theorem (finite case) [7]). *Let G be a finite graph. Let real numbers $k_i \geq 0$, $i = 1, 2, \ldots, p$. Then there exists a finite optimal $L(k_1, k_2, \ldots, k_p)$-labeling $f^* : V(G) \to [0, \infty)$ in which the smallest label is 0 and all labels belong to the set $D(k_1, k_2, \ldots, k_p)$. Hence, $\lambda(G; k_1, k_2, \ldots, k_p)$ belongs to $D(k_1, k_2, \ldots, k_p)$. Additionally, if G is a finite graph, then for each label and for the span of f^*, the sum of coefficients $\sum_i a_i$ is less than the number of vertices.*

Due to the D-set Theorem, previous optimal integer labeling results are compatible with the theory of real number labeling. An important property that is manifest in the setting of real number labelings is scaling:

Proposition 1.2 (Scaling Property). *For real numbers $d, k_i \geq 0$, $i = 1, 2, \ldots, p$,*

$$\lambda(G; d \cdot k_1, d \cdot k_2, \ldots, d \cdot k_p) = d \cdot \lambda(G; k_1, k_2, \ldots, k_p).$$

For any fixed p and any finite graph G it is proven in [7] that $\lambda(G; k_1, k_2, \ldots, k_p)$ is a continuous piecewise linear function of the real numbers k_i, where the pieces have nonnegative integer coefficients and where there are only finitely many pieces. The continuity means that it suffices to determine $\lambda(G; k_1, k_2, \ldots, k_p)$ for rational k_i's, and by scaling, it is enough to determine it for integer k_i's, which is the setting in which other researchers have worked. Indeed, our proofs here often reduce to the integer case. However, the analysis is more clear with the real number model, and more results have emerged by considering real number labelings. The theoretical results mentioned above give us additional tools to work with.

In particular, scaling implies that for $k_2 > 0$, $\lambda(G, k_1, k_2) = k_2 \lambda(G; k, 1)$, where $k = k_1/k_2$. This reduces the two-parameter function to a one parameter function, $\lambda(G; k, 1)$, $k \geq 0$. The results above ensure that this is a continuous, nondecreasing, piecewise linear function of k with finitely many pieces. Further, each piece has the form $ak + b$ for some integers $a, b \geq 0$. It is this function that we will obtain for (1) Paths P_n on n vertices; (2) Cycles C_n on $n \geq 3$ vertices; and (3) Wheels W_n, $n \geq 3$, consisting of a cycle C_n and a vertex adjacent to all of its vertices (see Figure 3).

The frequency channel separations k_i for two transmitters are often inversely proportional to the distance i between them [2]. Most articles assume that the separations are nonincreasing, $k_1 \geq k_2 \geq \ldots \geq k_p$. But this is not required in our theory, and there are different settings in which these labelings are a good model, but without the added assumption on the separations k_i (see [8]).

For additional motivation for studying paths and cycles, we mention these network models:

(a) A typical n-cell linear highway cellular system [1] along a highway (with the base-stations/transmitters in the center of each cell) can be modeled by path a P_n.

(b) A loop cellular system around a big city [1, 2], due to the high buildings, can be modeled by a cycle C_n.

Also, paths and cycles are induced subgraphs of many graphs, such as those that occur in typical cellular systems.

2 Results on Paths, Cycles, Wheels.

In their original paper on distance labeling, Griggs and Yeh [9] worked out the $(2,1)$-labeling number for paths and cycles, showing that $\lambda(C_n; 2, 1) = 4$ for all n, which is the same value as $\lambda(P_n; 2, 1)$ for all $n \geq 5$.

While working out the basics of the general theory of real number labeling with distance conditions, the authors determined $\lambda(P_n; k, 1)$ and $\lambda(C_n; k, 1)$ for arbitrary real $k \geq 0$. This was in 2003. But it took longer to work out the basics of the theory, which needed to be in the first paper of the series, which has just been accepted for publication [7]. In reviewing past work on integer labelings with distance conditions, the authors discovered that Georges and Mauro [4] had already in 1995 determined the values $\lambda(P_n; k_1, k_2)$ and $\lambda(C_n; k_1, k_2)$ for integers $k_1 \geq k_2$. Their formulas are given in terms of the ratio k_1/k_2; In terms of the theory of real number labelings, we see clearly why this is the case. For by using scaling (and factoring out k_2), their formulas are equivalent to giving $\lambda(P_n; \frac{k_1}{k_2}, 1)$ and $\lambda(C_n; \frac{k_1}{k_2}, 1)$. That is, letting $k = \frac{k_1}{k_2}$, we obtain $\lambda(P_n; k, 1)$ and $\lambda(C_n; k, 1)$ for rational $k \geq 1$. Next, by continuity, we deduce $\lambda(P_n; k, 1)$ and $\lambda(C_n; k, 1)$ for *real* $k \geq 1$.

Thus, our formulas for paths and cycles can be deduced for $k \geq 1$ from the existing theory. We feel our main contribution is to place them in the context of real number labelings, where the formulas are more illuminating. Because our proofs use different methods, we present them in detail. In addition, we expand the results to cover $k < 1$, which had not been done before. Finally, we also treat the wheels W_n. Here are our results:

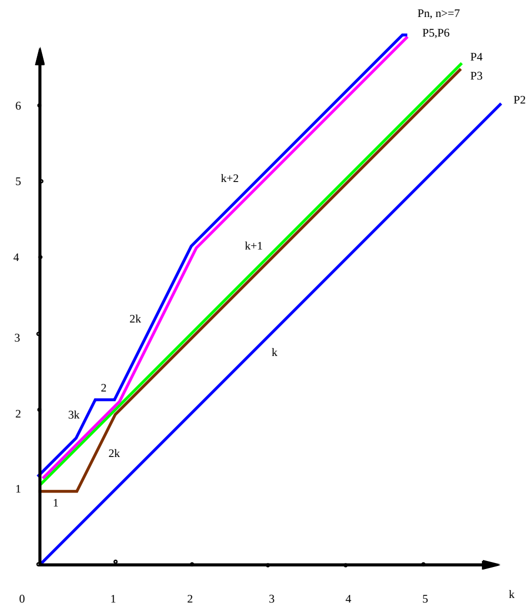


Figure 1: The minimum span $\lambda(P_n; k, 1)$

Theorem 2.1 (Paths Theorem). *For real $k \geq 0$, we have*

$\lambda(P_2; k, 1) = k$.

$$\lambda(P_3; k, 1) = \begin{cases} 1 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 2k & \text{if } \frac{1}{2} \leq k \leq 1 \\ k+1 & \text{if } k \geq 1 \end{cases}$$

$\lambda(P_4; k, 1) = k + 1$.

$$\lambda(P_5; k, 1) = \lambda(P_6; k, 1) = \begin{cases} k+1 & \text{if } 0 \leq k \leq 1 \\ 2k & \text{if } 1 \leq k \leq 2 \\ k+2 & \text{if } k \geq 2 \end{cases}$$

For $n \geq 7$,

$$\lambda(P_n; k, 1) = \begin{cases} k+1 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 3k & \text{if } \frac{1}{2} \leq k \leq \frac{2}{3} \\ 2 & \text{if } \frac{2}{3} \leq k \leq 1 \\ 2k & \text{if } 1 \leq k \leq 2 \\ k+2 & \text{if } k \geq 2 \end{cases}$$

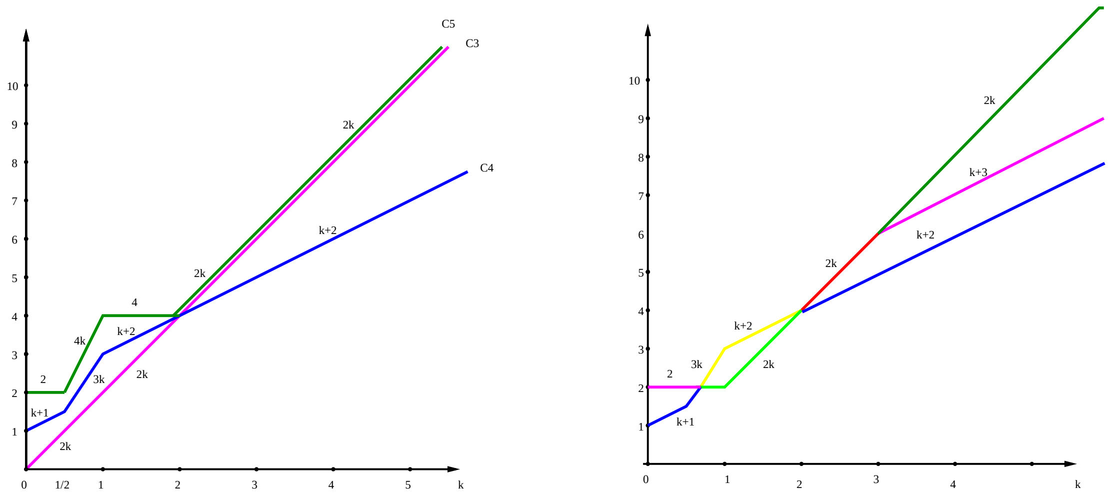


Figure 2: The minimum span $\lambda(C_n; k, 1)$ for $n = 3, 4, 5$ and $n \geq 6$.

Theorem 2.2 (Cycles Theorem)**.** *For real $k \geq 0$, we have*

$\lambda(C_3; k, 1) = 2k.$

$$\lambda(C_4; k, 1) = \begin{cases} k+1 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 3k & \text{if } \frac{1}{2} \leq k \leq 1 \\ k+2 & \text{if } k \geq 1 \end{cases}$$

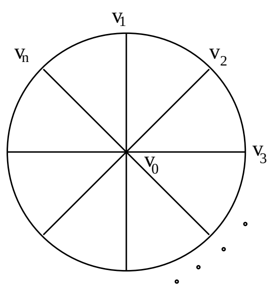
$$\lambda(C_5; k, 1) = \begin{cases} 2 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 4k & \text{if } \frac{1}{2} \leq k \leq 1 \\ 4 & \text{if } 1 \leq k \leq 2 \\ 2k & \text{if } k \geq 2 \end{cases}$$

For $n \geq 6$, $\lambda(C_n, k, 1)$:

$\lambda(C_n; k, 1)$	$n \equiv 0 \pmod 4$	$n \not\equiv 0 \pmod 4$
if $0 \leq k \leq \frac{1}{2}$	$k+1$	2
if $\frac{1}{2} \leq k \leq \frac{2}{3}$	$3k$	2

$\lambda(C_n; k, 1)$	$n \equiv 0 \pmod 3$	$n \not\equiv 0 \pmod 3$
if $\frac{2}{3} \leq k \leq 1$	2	$3k$
if $1 \leq k \leq 2$	$2k$	$k+2$

$\lambda(C_n; k, 1)$	$n \equiv 0 \pmod 4$	$n \equiv 2 \pmod 4$	$n \equiv 1$ or $3 \pmod 4$
if $2 \leq k \leq 3$	$k+2$	$2k$	$2k$
if $k \geq 3$	$k+2$	$k+3$	$2k$



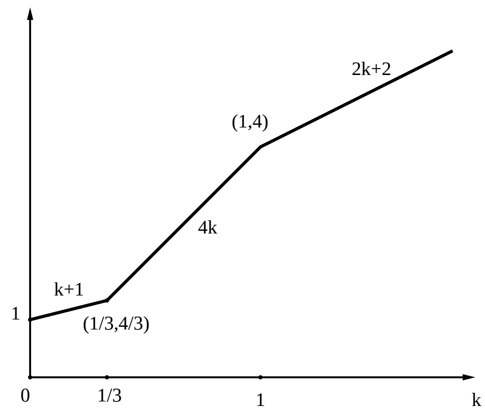


Figure 3: Wheel W_n, $n \geq 3$ (left), and $\lambda(W_4; k, 1)$ (right).

Theorem 2.3 (Wheels Theorem). *For real $k \geq 0$, we have*

$\lambda(W_3; k, 1) = 3k$.

$$\lambda(W_4; k, 1) = \begin{cases} k+1 & \text{if } 0 \leq k \leq \frac{1}{3} \\ 4k & \text{if } \frac{1}{3} \leq k \leq 1 \\ 2k+2 & \text{if } k \geq 1 \end{cases}$$

For odd $n \geq 5$,

$$\lambda(W_n; k, 1) = \begin{cases} \frac{n-1}{2} & \text{if } 0 \leq k \leq \frac{1}{3} \\ 3k + \frac{n-3}{2} & \text{if } \frac{1}{3} \leq k \leq \frac{1}{2} \\ nk & \text{if } \frac{1}{2} \leq k \leq 1 \\ k+n-1 & \text{if } 1 \leq k \leq \frac{n-1}{2} \\ 3k & \text{if } k \geq \frac{n-1}{2} \end{cases}$$

For even $n \geq 5$,

$$\lambda(W_n; k, 1) = \begin{cases} k + \frac{n}{2} - 1 & \text{if } 0 \leq k \leq \frac{1}{3} \\ 4k + \frac{n}{2} - 2 & \text{if } \frac{1}{3} \leq k \leq \frac{1}{2} \\ nk & \text{if } \frac{1}{2} \leq k \leq 1 \\ k+n-1 & \text{if } 1 \leq k \leq \frac{n}{2} - 1 \\ 2k + \frac{n}{2} & \text{if } k \geq \frac{n}{2} - 1 \end{cases}$$

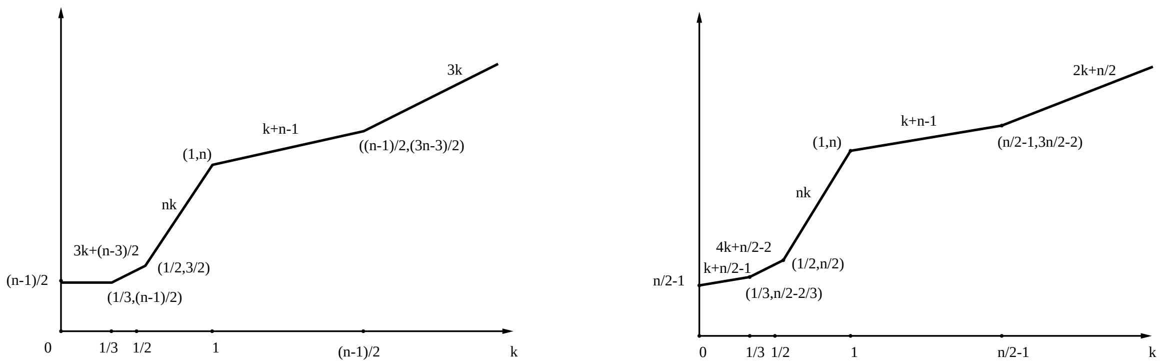


Figure 4: $\lambda(W_n; k, 1)$ for odd $n \geq 5$ (left), and for even $n \geq 6$ (right).

3 The Proof for Paths.

Let the vertices of path P_n be called, in order starting from one end, $v_1, \ldots, v_n$. The result is immediate for P_2, which is a single edge. Next consider P_3.

Proposition 3.1. *For real $k \geq 0$, we have*

$$\lambda(P_3; k, 1) = \begin{cases} 1 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 2k & \text{if } \frac{1}{2} \leq k \leq 1 \\ k+1 & \text{if } k \geq 1 \end{cases}$$

Proof:

The upper bound is attained by labelling f with

$$(f(v_1), f(v_2), f(v_3)) = \begin{cases} (0, k, 1) & \text{if } 0 \leq k \leq \frac{1}{2} \\ (0, k, 2k) & \text{if } \frac{1}{2} \leq k \leq 1 \\ (0, k+1, 1) & \text{if } k \geq 1 \end{cases}$$

We will show that is the lower bound, too.

By the condition at distance two, $f(v_1)$ or $f(v_3) \geq 1$. Hence, $\lambda(P_3; k, 1) \geq 1$, which gives the desired bound for $0 \leq k \leq \frac{1}{2}$.

Claim 1. For $\frac{1}{2} \leq k \leq 1$, $\lambda(P_3; k, 1) \geq 2k$.
Proof of Claim 1: The labels for any two of the three vertices of P_3 must differ by at least k in this range, and so the span of any labeling must be at least $2k$.$\square$

Claim 2. For $k \geq 1$, $\lambda(P_3; k, 1) \geq k + 1$.
Proof of Claim 2: Assume $\lambda(P_3; k, 1) = l < k + 1$. By the D-Set Theorem, there is an optimal labelling $f \in L(k, 1)(P_3)$ with span$(f) = l < 2k$. We may assume that at least two of the three labels are $< k$ (otherwise, we may replace $f(v)$ by $l - f(v)$ for all vertices v). By the distance conditions, labels $< k$ cannot be adjacent, and so $f(v_1), f(v_3) < k$. By the condition at distance two, $f(v_1)$ or $f(v_3) \geq 1$. Hence, $f(v_2) \geq k+1$, which contradicts the assumption .$\square$ $\blacksquare$

Proposition 3.2. *For real $k \geq 0$, we have*
$\lambda(P_4; k, 1) = k + 1.$

Proof: The upper bound is attained by labelling$(f(v_1), f(v_2), f(v_3), f(v_4)) = (k, 0, k+1, 1)$. We will show that is also the lower bound.

For $k \geq 1$ we have $\lambda(P_4; k, 1) \geq \lambda(P_3; k, 1) = k + 1$, as desired. It remains to treat small k. It suffices to prove the lower bound of $k + 1$ for $0 < k < 1$, since it follows at $k = 0$ by continuity of λ.

Claim 3. For $0 < k < 1$, we have $\lambda(P_4; k, 1) \geq k + 1$.
Proof of Claim 3: Assume to the contrary that for some such k, $l = \lambda(P_4; k, 1) < k+1$, and let f be an optimal labeling as in the D-Set Theorem.

Suppose $f(v_2) < 1$. If $f(v_2) = 0$, then by the distance conditions, both $f(v_1), f(v_3) \geq k$, and the larger of the two must then be $\geq k + 1$, contradicting the assumption on l. So $f(v_2)$ must of the form ik for some integer $i > 0$ (since it is in $D(k, 1)$ and is < 1). Then $f(v_4)$ must be at least $ik + 1 \geq k + 1$, again contradicting the assumption.

Hence, $f(v_2) \geq 1$. Now we define a complementary labeling f' by $f'(v) = l - f(v)$. While f' is also an optimal $L(k, 1)$-labeling, it may not be one as in the D-Set Theorem (with all labels in $D(k, 1)$). But we can obtain such a labeling, call it f'', as in the proof of the D-Set Theorem [7]: For each v let $f''(v)$ be the largest element of $D(k, 1)$ that is $\leq f'(v)$. Now we have that $f''(v) \leq f'(v) = l - f(v) \leq l - 1 < k < 1$, and we get a contradiction the same way as before. □ ■

Proposition 3.3. *For real $k \geq 0$, we have*

$$\lambda(P_5; k, 1) = \lambda(P_6; k, 1) = \begin{cases} k+1 & \text{if } 0 \leq k \leq 1 \\ 2k & \text{if } 1 \leq k \leq 2 \\ k+2 & \text{if } k \geq 2 \end{cases}$$

Proof: Both graphs have the same spans. For the upper bound, it is enough to show how to label P_6:

$$\begin{cases} (k+1, k, 0, k+1, 1, 0) & \text{if } 0 \leq k \leq 1 \\ (0, k, 2k, 0, k, 2k) & \text{if } 1 \leq k \leq 2 \\ (0, k+1, 1, k+2, 0, k+1) & \text{if } k \geq 2 \end{cases}$$

It remains to prove the lower bound on P_5. For $0 \leq k \leq 1$ we use $\lambda(P_5; k, 1) \geq \lambda(P_4; k, 1) = k + 1$. Next consider k between 1 and 2:

Claim 4. For $1 \leq k \leq 2$, $\lambda(P_5; k, 1) \geq 2k$.
Proof of Claim 4: Assume $l = \lambda(P_5; k, 1) < 2k$, and let f be an optimal labeling as in the D-Set Theorem. We may assume that at least two of the three labels $f(v_2), f(v_3), f(v_4)$ are $< k$ (or else take a complementary labeling f'' as in the proof of Claim 3). By the distance conditions, these two labels cannot be adjacent, and so $f(v_2)$ and $f(v_4)$ are both $< k$ and at least 1 apart. The larger of the two labels, say it is $f(v_2)$, then satisfies $1 \leq f(v_2) < k$. So $f(v_1), f(v_3) \geq k + 1$. By the condition at distance two, $f(v_1)$ or $f(v_3) \geq k + 2 \geq 2k$, contradicting $l < 2k$.□

Claim 5. For $k \geq 2$, $\lambda(P_5; k, 1) \geq k + 2$.
Proof of Claim 5: Assume to the contrary that $l = \lambda(P_5; k, 1) < k + 2$, and let f be an optimal labeling as in the D-Set Theorem. Following the proof of Claim 4 again leads to $f(v_1)$ or $f(v_3) \geq k + 2$, a contradiction. □ ■

Proposition 3.4. *Let $n \geq 7$. For real $k \geq 0$, we have*

$$\lambda(P_n; k, 1) = \begin{cases} k+1 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 3k & \text{if } \frac{1}{2} \leq k \leq \frac{2}{3} \\ 2 & \text{if } \frac{2}{3} \leq k \leq 1 \\ 2k & \text{if } 1 \leq k \leq 2 \\ k+2 & \text{if } k \geq 2 \end{cases}$$

Proof: According to the value of k, we repeat an underlined pattern until all of P_n is labeled to achieve the stated optimal spans:

$$\begin{cases} (\underline{0, k+1, 1, k}, \ldots) & \text{if } 0 \leq k \leq \frac{1}{2} \\ (\underline{0, k, 2k, 3k}, \ldots) & \text{if } \frac{1}{2} \leq k \leq \frac{2}{3} \\ (\underline{0, 1, 2}, \ldots) & \text{if } \frac{2}{3} \leq k \leq 1 \\ (\underline{0, k, 2k}, \ldots) & \text{if } 1 \leq k \leq 2 \\ (\underline{0, k+1, 1, k+2}, \ldots) & \text{if } k \geq 2 \end{cases}$$

The lower bounds follow from those for P_5, except in the range $\frac{1}{2} < k < 1$. Next, one can easily check that for $\frac{1}{2} < k \leq \frac{2}{3}$, $(k + 1, 3k) \cap D(k, 1) = \emptyset$, and for $\frac{2}{3} < k < 1$, $(k + 1, 2) \cap D(k, 1) = \emptyset$. By the D-Set Theorem, it then suffices to prove:
Claim 6. For $\frac{1}{2} < k < 1$, $\lambda(P_7; k, 1) > k + 1$.
Proof of Claim 6: Assume for some such k that $l = \lambda(P_7; k, 1) \leq k + 1$, and let f be an optimal labeling as in the D-Set Theorem. We may assume that at least two of the three labels $f(v_3), f(v_4), f(v_5)$ are < 1 (or else take a complementary labeling f'' as in the proof of Claim 3). These two labels cannot be at distance two, so we may assume they are at v_3 and v_4 (or else reverse the order of the vertices on P_7). We only need to work now on v_1 through v_6. By symmetry, we may assume $f(v_3) > f(v_4)$, so that $f(v_3) \geq k$.

Since $f(v_3) < 1$, the condition at distance two forces $f(v_1) \geq f(v_3) + 1 \geq k + 1$. Due to the span of f, it must be that $f(v_1) = k + 1$, which forces $f(v_3) = k$ and $f(v_4) = 0$. Then v_1 forces $f(v_2) \leq 1$, while v_3 and v_4 force $f(v_2) > 1$, a contradiction. □ ■

This completes the proof of Theorem 2.1.

4 The Proof for Cycles.

Let the vertices of cycle C_n be called, in order going around, $v_1, \ldots, v_n$. The result is almost immediate for C_3, for which the optimal labeling is $(0, k, 2k)$. We take care of C_4 and C_5 in the next two propositions.

Proposition 4.1. *For real $k \geq 0$, we have*

$$\lambda(C_4; k, 1) = \begin{cases} k+1 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 3k & \text{if } \frac{1}{2} \leq k \leq 1 \\ k+2 & \text{if } k \geq 1 \end{cases}$$

Proof: The upper bound is attained by the labeling

$$(f(v_1), f(v_2), f(v_3), f(v_4)) = \begin{cases} (0, k, 1, k+1) & \text{if } 0 \le k \le \frac{1}{2} \\ (0, k, 2k, 3k) & \text{if } \frac{1}{2} \le k \le 1 \\ (0, k+1, 1, k+2) & \text{if } k \ge 1 \end{cases}$$

For the lower bound, consider an optimal labeling f as described by the D-Set Theorem. Suppose $f(v_1) = 0$. The distance conditions force $f(v_2), f(v_3), f(v_4) \ge \min\{k, 1\}$. The span among $\{f(v_2), f(v_3), f(v_4)\}$ is $\ge \lambda(P_3; k, 1)$, and so $\lambda(C_4; k, 1) \ge \lambda(P_3; k, 1) + \min\{k, 1\}$. Applying Theorem 2.1 to evaluate this, we obtain the stated bound. ■

Proposition 4.2. *For real $k \ge 0$, we have*

$$\lambda(C_5; k, 1) = \begin{cases} 2 & \text{if } 0 \le k \le \frac{1}{2} \\ 4k & \text{if } \frac{1}{2} \le k \le 1 \\ 4 & \text{if } 1 \le k \le 2 \\ 2k & \text{if } k \ge 2 \end{cases}$$

Proof:

The upper bound is attained by labeling

$$\begin{cases} (0, k, 1, k+1, 2) & \text{if } 0 \le k \le \frac{1}{2} \\ (0, k, 2k, 3k, 4k) & \text{if } \frac{1}{2} \le k \le 1 \\ (0, 2, 4, 1, 3) & \text{if } 1 \le k \le 2 \\ (0, k, 2k, 1, k+1) & \text{if } k \ge 2 \end{cases}$$

We will show that is the lower bound too.

Claim 1. For $0 \le k \le \frac{1}{2}$, we have $\lambda(C_5; k, 1) \ge 2$.
Proof of Claim 1: Let f be an optimal labeling of C_5 as in the D-Set Theorem. For any three vertices of C_5, some two are at distance two, so at most two vertices have labels $f(v)$ in $[0, 1)$. Similarly, at most two vertices have labels in $[1, 2)$. So some vertex must have label ≥ 2, and the span of f is ≥ 2. □

Claim 2. For $\frac{1}{2} \le k \le 1$, we have $\lambda(C_5; k, 1) \ge 4k$, and for $1 \le k \le 2$, we have $\lambda(C_5; k, 1) \ge 4$.
Proof of Claim 2: Suppose $\frac{1}{2} \le k \le 1$. If f is an optimal labeling as in the D-Set Theorem, then the distance conditions imply that any two labels differ by at least k, and so the span of f is at least $4k$, as claimed. For $k \ge 1$, we then have $\lambda(C_5; k, 1) \ge \lambda(C_5; 1, 1) = 4$. □

Claim 3. For $k \ge 2$, we have $\lambda(C_5; k, 1) \ge 2k$.
Proof of Claim 3: Let f be an optimal labeling of C_5 as in the D-Set Theorem. Of any three vertices, some two are adjacent. Thus, the conditions at distance one imply that at most two vertices have labels $f(v)$ in $[0, k)$. Similarly, at most two vertices have labels in $[k, 2k)$. So some vertex must have label $\ge 2k$, and the span of f is $\ge 2k$. □■

It remains to treat C_n for all $n \ge 6$. The first proposition gives labelings that achieve the stated bounds.

Proposition 4.3. *Let $n \ge 6$. For real $k \ge 0$, there are labelings that achieve the bounds in the Theorem.*

Proof: Here are the labelings, depending on the value of k and on n mod 3 and mod 4. The underlined sections are repeated as many times as needed.

$$\text{For } 0 \le k \le \tfrac{1}{2}\text{:}\quad \begin{cases} (\underline{0, k, 1, k+1}) & \text{if } n \equiv 0 \pmod 4 \\ (0, 1, 2, 0, 1, 2, 0, 1, 2, \underline{0, k, 1, 2}) & \text{if } n \equiv 1 \pmod 4 \\ (0, 1, 2, 0, 1, 2, \underline{0, k, 1, 2}) & \text{if } n \equiv 2 \pmod 4 \\ (0, 1, 2, \underline{0, k, 1, 2}) & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

$$\text{For } \tfrac{1}{2} \le k \le \tfrac{2}{3}\text{:}\quad \begin{cases} (\underline{0, k, 2k, 3k}) & \text{if } n \equiv 0 \pmod 4 \\ (0, 1, 2, 0, 1, 2, 0, 1, 2, \underline{0, k, 2k, 2}) & \text{if } n \equiv 1 \pmod 4 \\ (0, 1, 2, 0, 1, 2, \underline{0, k, 2k, 2}) & \text{if } n \equiv 2 \pmod 4 \\ (0, 1, 2, \underline{0, k, 2k, 2}) & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

$$\text{For } \tfrac{2}{3} \le k \le 1\text{:}\quad \begin{cases} (\underline{0, 1, 2}) & \text{if } n \equiv 0 \pmod 3 \\ (0, k, 2k, 3k, \underline{0, 1, 2}) & \text{if } n \equiv 1 \pmod 3 \\ (0, k, 2k, 3k, 0, k, 2k, 3k, \underline{0, 1, 2}) & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

$$\text{For } 1 \le k \le 2\text{:}\quad \begin{cases} (\underline{0, k, 2k}) & \text{if } n \equiv 0 \pmod 3 \\ (0, k+1, 1, k+2, \underline{0, k, k+2}) & \text{if } n \equiv 1 \pmod 3 \\ (0, k+1, 1, k+2, 0, k+1, 1, k+2, \underline{0, k, k+2}) & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

$$\text{For } 2 \le k \le 3\text{:}\quad \begin{cases} (\underline{0, k+1, 1, k+2}) & \text{if } n \equiv 0 \pmod 4 \\ (0, k, 2k, 0, k, 2k, 0, k, 2k, \underline{0, k+1, 1, k+2}) & \text{if } n \equiv 1 \pmod 4 \\ (0, k, 2k, 0, k, 2k, \underline{0, k+1, 1, k+2}) & \text{if } n \equiv 2 \pmod 4 \\ (0, k, 2k, \underline{0, k+1, 1, k+2}) & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

$$\text{For } k \ge 3\text{:}\quad \begin{cases} (\underline{0, k+1, 1, k+2}) & \text{if } n \equiv 0 \pmod 4 \\ (0, k, 2k, 0, k, 2k, 0, k, 2k, \underline{0, k+1, 1, k+2}) & \text{if } n \equiv 1 \pmod 4 \\ (0, k+1, 1, k+2, 2, k+3, \underline{0, k+1, 1, k+2}) & \text{if } n \equiv 2 \pmod 4 \\ (0, k, 2k, \underline{0, k+1, 1, k+2}) & \text{if } n \equiv 3 \pmod 4 \end{cases} \qquad \blacksquare$$

It remains to prove the lower bounds for $n \ge 7$. We begin with $k \le \frac{2}{3}$, which splits into cases according to $n \pmod 4$. If $n \equiv 0 \pmod 4$, then $n \ge 8$, and C_n contains an induced P_7. Thus, $\lambda(C_n; k, 1) \ge \lambda(P_7; k, 1)$, which is the desired formula, $k + 1$ for $0 \le k \le \frac{1}{2}$ and $3k$ for $\frac{1}{2} \le k \le \frac{2}{3}$. We now treat the other values of n:

Proposition 4.4. *Let $n \ge 6$ with $n \not\equiv 0 \pmod 4$. For $0 \le k \le \frac{2}{3}$, the spans stated in the Theorem cannot be improved.*

Proof: It suffices to prove the lower bound, 2, at $k = 0$. Assume to the contrary that for some such n, $l = \lambda(C_n; 0, 1) < 2$, and let f be an optimal labeling. Then each vertex v_i has its label $f(v_i)$ either in the interval $[0, 1)$ or else in $[1, 2)$. Since no two vertices at distance two can have labels in the same unit interval, it must be that the labels going around C_n have two in $[0, 1)$ followed by two in $[1, 2)$ followed by two in $[0, 1)$ again, and so on. But this is possible only if 4 divides n, a contradiction. $\blacksquare$

We next treat $\frac{2}{3} \le k \le 1$. If $n \ge 8$, then C_n contains an induced P_7 as above, and the value of $\lambda(P_7; k, 1) = 2$ is a lower bound. For $n \equiv 0 \pmod 3$, this is the desired bound, though we must still prove this bound for C_6: The vertices v_1, v_3, v_5 in C_6 are pairwise at distance two, so require span at least 2. This still leaves $n \not\equiv 0 \pmod 3$ for this range in k:

Proposition 4.5. *Let $n \geq 7$ with $n \not\equiv 0 \pmod 3$. For $\frac{2}{3} \leq k \leq 1$, we have that $\lambda(C_n; k, 1) \geq 3k$.*

Proof: Assume to the contrary that for some such n and k we have $\lambda(C_n; k, 1) < 3k$, and let f be an optimal labeling as in the D-Set Theorem. Every label $f(v_i)$ is in one of the intervals $[0, k), [k, 2k), [2k, 3k)$, and no two labels within distance two are in the same interval. Hence, each of these intervals contains labels for at most $\lfloor n/3 \rfloor$ of the vertices. But this is less than $n/3$, since $n \not\equiv 0 \pmod 3$, so we have not accounted for all n vertices, a contradiction. ■

The next range up is $1 \leq k \leq 2$. The lower bound here for cycles C_n, $n \geq 6$ with $n \equiv 0 \pmod 3$, follows immediately from the span of P_5, which is $2k$ in this range. We then need to treat the remaining n:

Proposition 4.6. *Let $n \geq 6$ with $n \not\equiv 0 \pmod 3$. For $1 \leq k \leq 2$, we have that $\lambda(C_n; k, 1) \geq k + 2$.*

Proof: For such an n and k, let f be an optimal labeling as in the D-Set Theorem. No two labels < 1 can be within distance two of each other, so there are at most $\lfloor n/3 \rfloor < n/3$ such labels. Thus, there exist some three consecutive vertices with labels ≥ 1. Using the fact that $\lambda(P_3; k, 1) = k + 1$, we get that the span of f is at least $k + 2$. ■

We now treat the large values, $k \geq 2$. Again, we get a lower bound from the span of P_5, which is $k + 2$ in this range. It is the bound we want for $n \equiv 0 \pmod 4$. For other $n \geq 6$ we must do better.

Proposition 4.7. *Let $n \geq 6$ with $n \not\equiv 0 \pmod 4$. If n is odd, then we have $\lambda(C_n; k, 1) \geq 2k$ for $k \geq 2$. If n is even, then we have $\lambda(C_n; k, 1) \geq 2k$ for $2 \leq k \leq 3$ and $\lambda(C_n; k, 1) \geq k + 3$ for $k \geq 3$.*

Proof: First suppose $n \geq 6$ is odd, and $k \geq 2$. Let f be an optimal labeling as in the D-Set Theorem. No two adjacent vertices have labels less than k apart, so the number of vertices with labels in $[0, k)$ is at most $\lfloor n/2 \rfloor < n/2$, and the same is true for labels in $[k, 2k)$. Hence, some vertex has label $\geq 2k$, and so the span of f is at least $2k$.

Then suppose $n \geq 6$ is even, but $\not\equiv 0 \pmod 4$, say $n = 2r$ where r is odd. We must show that $\lambda(C_n; k, 1) \geq \min\{2k, k + 3\}$ for $k \geq 2$. Suppose not, say $l = \lambda(C_n; k, 1) < \min\{2k, k + 3\}$ for some k, and let f be an optimal labeling as in the D-Set Theorem. Arguing as for odd n, we find that there are $n/2$ labels each in the intervals $[0, k)$ and $[k, l]$ (as $l < 2k$), and they alternate between the two intervals. Looking at the r labels in $[0, k)$ in order going around, we find that consecutive ones, which are at distance two in C_n, differ by at least one. Since r is odd, some label in $[0, k)$ is at least 2. Its two neighbors on C_n have "large labels" (at least k). The distance conditions mean that each neighbor has label at least $k + 2$, and the larger of the two must then be at least $k + 3$. So the span of f is at least $k + 3$, which contradicts the assumption on l. ■

This completes the proof of Theorem 2.2.

5 The Proof for Wheels

While the wheel W_n is closely related to the cycle C_n, note that the extra vertex brings the diameter down to just two, which clearly affects distance labelings. In fact, for an $L(k,1)$-labeling, any two of the $n+1$ labels must differ by at least $\min\{k,1\}$.

As before, we denote the vertices going around the n-cycle in C_n by $v_1,\ldots,v_n$. We denote the extra vertex adjacent to the cycle by v_0. Since W_3 is just the complete graph K_4, it has optimal span $3k$. Next we consider W_4:

Proposition 5.1. *For real $k \geq 0$, we have*

$$\lambda(W_4;k,1) = \begin{cases} k+1 & \text{if } 0 \leq k \leq \frac{1}{3} \\ 4k & \text{if } \frac{1}{3} \leq k \leq 1 \\ 2k+2 & \text{if } k \geq 1 \end{cases}$$

Proof: The upper bound is attained by the following labelings f, in which we give the label of the central vertex, $f(v_0)$, first followed after a semicolon by the labels going around the n-cycle:

$$\begin{cases} (2k;0,k,1,k+1) & \text{if } 0 \leq k \leq \frac{1}{3} \\ (2k;0,k,3k,4k) & \text{if } \frac{1}{3} \leq k \leq 1 \\ (0;k,2k+1,k+1,2k+2) & \text{if } k \geq 1 \end{cases}$$

We must verify this is also a lower bound for all k. For $0 \leq k \leq \frac{1}{3}$, we have $\lambda(W_4;k,1) \geq \lambda(C_4;k,1) = k+1$.

For $\frac{1}{3} \leq k \leq 1$, any two labels must differ by at least k, so that the span of an $L(k,1)$-labeling is at least $4k$. It remains to treat large k:

Claim 1. For $k \geq 1$, $\lambda(W_4;k,1) \geq 2k+2$.

Proof of Claim 1: Let f be an optimal labeling as in the D-Set Theorem. Any three vertices in W_4 induce either a path P_3, which has span $k+1$, or a cycle C_3, which has span $2k \geq k+1$. So at most two vertices have labels in the interval $[0,k+1)$, and at most two have labels in $[k+1,2k+2)$. Hence, some label is at least $2k+2$, and so is the span of f. $\square$ $\blacksquare$

For $n \geq 5$, we split according to whether it is odd or even.

Proposition 5.2. *Let n be an odd integer ≥ 5. For real $k \geq 0$, we have*

$$\lambda(W_n;k,1) = \begin{cases} \frac{n-1}{2} & \text{if } 0 \leq k \leq \frac{1}{3} \\ 3k+\frac{n-3}{2} & \text{if } \frac{1}{3} \leq k \leq \frac{1}{2} \\ nk & \text{if } \frac{1}{2} \leq k \leq 1 \\ k+n-1 & \text{if } 1 \leq k \leq \frac{n-1}{2} \\ 3k & \text{if } k \geq \frac{n-1}{2} \end{cases}$$

Proof: The upper bound is attained by these labelings f, in which $f(v_0)$ is listed first:

For $0 \leq k \leq \frac{1}{3}$: $(2k;0,k,1,k+1,2,k+2,3,k+3,\ldots,\frac{n-3}{2},k+\frac{n-3}{2},\frac{n-1}{2})$

For $\frac{1}{3} \leq k \leq \frac{1}{2}$: $(2k;0,k,3k,4k,3k+1,4k+1,3k+2,4k+2,\ldots,4k+\frac{n-5}{2},3k+\frac{n-3}{2})$

For $\frac{1}{2} \leq k \leq 1$: $(0;k,2k,3k,\ldots,nk)$

For $1 \leq k \leq \frac{n-1}{2}$:
$(0; k, k+\frac{n+1}{2}, k+1, k+\frac{n+3}{2}, k+2, k+\frac{n+5}{2}, \ldots, k+\frac{n-3}{2}, k+n-1, k+\frac{n-1}{2})$
For $k \geq \frac{n-1}{2}$:
$(0; k, 2k+1, k+1, 2k+2, k+2, 2k+3, \ldots, k+\frac{n-5}{2}, 2k+\frac{n-3}{2}, k+\frac{n-3}{2}, 3k, 2k)$.

We need to verify that these values are lower bounds. We begin with small k:

Claim 2. Let $n \geq 5$ be odd. For $0 \leq k \leq \frac{1}{3}$, $\lambda(W_n; k, 1) \geq \frac{n-1}{2}$.
Proof of Claim 2: Let f be an optimal labeling with smallest label 0. Of the n vertices on the outer cycle, no three can be in the same interval $[i, i+1)$, since some two of any three vertices on the cycle are at distance two in W_n. Thus, some vertex on the cycle has label outside $[0, \frac{n-1}{2})$, and so the span of f is at least $\frac{n-1}{2}$. $\square$

Claim 3. Let $n \geq 5$ be odd. For $\frac{1}{3} \leq k \leq \frac{1}{2}$, $\lambda(W_n; k, 1) \geq 3k + \frac{n-3}{2}$.
Proof of Claim 3: Let f be an optimal labeling as in the D-Set Theorem. Using the distance conditions, all vertices with labels in the interval $[0, 1)$ are mutually adjacent. So there are at most three of them, and, if there are three, one of them must be the center, v_0 (and $k < 1/2$). The same is true for all of the intervals $I_i := [i, i+1)$, $0 \leq i \leq \frac{n-3}{2}$. This means we can have at most n vertices with labels in $[0, \frac{n-1}{2})$, so that some vertex w has label $\geq \frac{n-1}{2}$. If there are two such vertices, the largest label is $\geq k + \frac{n-1}{2} \geq 3k + \frac{n-3}{2}$, the desired bound. If there is only one such vertex, we must look more closely: Some interval $[j, j+1)$ must contain labels for three vertices (one of which is v_0), while the others have just two each. The largest label in $[j, j+1)$ is at least $2k + j$. The two labels in $[j+1, j+2)$ are then at least k larger, $3k + j$. The two labels in $[j+2, j+3)$ are at least $3k + j + 1$, because each represents a vertex in the n-cycle that is distance two from one or both vertices with labels in $[j+1, j+2)$. Repeating this idea, we eventually find that the label $f(w) \geq 3k + \frac{n-3}{2}$, as claimed. $\square$

For $\frac{1}{2} \leq k \leq 1$, the lower bound is easy: Since any two vertices have labels at least k apart (as $k \leq 1$), and there are $n+1$ vertices, the optimal span is at least nk.

For $1 \leq k \leq \frac{n-1}{2}$, any pair of labels in some optimal labeling f (as in the D-Set Theorem) differ by at least $\min\{k, 1\} = 1$. Suppose the labels are $0 = x_0 < x_1 < x_2 \cdots < x_n$. Since v_0 is adjacent to every other vertex, there exists i such that $x_{i+1} - x_i \geq k$, hence the span $x_n \geq k + n - 1$, as claimed.

Finally, we consider large k:

Claim 4. Let $n \geq 5$ be odd. For $k \geq \frac{n-1}{2}$, $\lambda(W_n; k, 1) \geq 3k$.
Proof of Claim 4: Let f be an optimal labeling with smallest label 0. Since W_n has diameter two, no two adjacent vertices have labels in $[0, k)$. That is, the vertices with labels in $[0, k)$ form an independent set, and the same is true for the intervals $[k, 2k)$ and $[2k, 3k)$. Since W_n has chromatic number 4, some vertex must have a larger label, which is $\geq 3k$, and so $\lambda(W_n; k, 1) \geq 3k$. $\square$ $\blacksquare$

Having completed the proof for odd n, it remains to treat even $n \geq 5$.

Proposition 5.3. *Let n be an even integer ≥ 5. For real $k \geq 0$, we have*

$$\lambda(W_n; k, 1) = \begin{cases} k + \frac{n}{2} - 1 & \textit{if } 0 \leq k \leq \frac{1}{3} \\ 4k + \frac{n}{2} - 2 & \textit{if } \frac{1}{3} \leq k \leq \frac{1}{2} \\ nk & \textit{if } \frac{1}{2} \leq k \leq 1 \\ k + n - 1 & \textit{if } 1 \leq k \leq \frac{n}{2} - 1 \\ 2k + \frac{n}{2} & \textit{if } k \geq \frac{n}{2} - 1 \end{cases}$$

Proof: The upper bound is attained by these labelings f, in which $f(v_0)$ is listed first:

For $0 \leq k \leq \frac{1}{3}$: $(2k; 0, k, 1, k+1, 2, k+2, 3, k+3, \ldots, \frac{n}{2} - 1, k + \frac{n}{2} - 1)$

For $\frac{1}{3} \leq k \leq \frac{1}{2}$: $(2k; 0, k, 3k, 4k, 3k+1, 4k+1, 3k+2, 4k+2, \ldots, 3k + \frac{n}{2} - 2, 4k + \frac{n}{2} - 2)$

For $\frac{1}{2} \leq k \leq 1$: $(0; k, 2k, 3k, \ldots, nk)$

For $1 \leq k \leq \frac{n-1}{2}$:
$(0; k, k + \frac{n}{2}, k+1, k + \frac{n}{2} + 1, k+2, k + \frac{n}{2} + 2, \ldots, k + \frac{n}{2} - 1, k + n - 1)$

For $k \geq \frac{n-1}{2}$:
$(0; k, 2k+1, k+1, 2k+2, k+2, 2k+3, \ldots, k + \frac{n}{2} - 1, 2k + \frac{n}{2})$.

We now prove the formulas above are lower bounds, beginning with small k:

Claim 5. Let $n \geq 5$ be even. For $0 \leq k \leq \frac{1}{3}$, $\lambda(W_n; k, 1) \geq k + \frac{n}{2} - 1$.
Proof of Claim 5: Let f be an optimal labeling of W_n for such n with smallest label 0. Because W_n has diameter two, at most one vertex v_i has label in $[0, k)$. If i is even, then all of the vertices in the independent set $I := \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ have labels at least k. If i is odd, instead use $I := \{v_2, v_4, v_6, \ldots, v_n\}$. Any two vertices in I are at distance 2, so their labels differ by at least one. Thus, the labels for I have span at least $|I| - 1$, and the span of f is at least $k + |I| - 1 = k + \frac{n}{2} - 1$. $\square$

Claim 6. Let $n \geq 5$ be even. For $\frac{1}{3} \leq k \leq \frac{1}{2}$, we have $\lambda(W_n; k, 1) \geq 4k + \frac{n}{2} - 2$.
Proof of Claim 6: Let f be an optimal labeling with smallest label 0. By the upper bound, f has span at most $\frac{n}{2}$ (note that $k \leq \frac{1}{2}$). Similar to the proof of Claim 3 above, we see that each interval $I_i := [i, i+1)$, $0 \leq i \leq \frac{n-2}{2}$, contains labels for a set of two mutually adjacent vertices, except that for one value j, I_j contains labels for three vertices, one of which is v_0. The largest of the three labels in I_j is at least $2k + j$.

If $j = \frac{n-2}{2}$, then we have a label that is at least $4k + \frac{n}{2} - 2$, the bound we seek.

On the other hand, suppose $j < \frac{n-2}{2}$. Then the two labels in I_{j+1} are at least k larger, so they are at least $3k + j$. We find successively (similar to the proof of Claim 3) that the two labels in I_{j+2} are at least $3k + j + 1$, and so on, and at least $3k + \frac{n-2}{2} - 1$ in $I_{\frac{n}{2}-2}$. The larger of the two labels in the last interval is then at least $4k + \frac{n}{2} - 2$, the desired bound. $\square$

For $\frac{1}{2} \leq k \leq \frac{n}{2} - 1$, the lower bounds follow by the same arguments as for Proposition 5.2. It remains to treat large k:

Claim 7. Let $n \geq 5$ be even. For $k \geq \frac{n}{2} - 1$, we have $\lambda(W_n; k, 1) \geq 2k + \frac{n}{2}$.
Proof of Claim 7: Let f be an optimal labeling of C_n with smallest label 0. The labels used by f are distinct and separated by at least one, except that $f(v_0)$ is separated by at

least k from the others. So if $f(v_0)$ is neither the smallest nor the largest label, then the span of f must be at least $2k + n - 2 \geq 2k + \frac{n}{2}$, which is the desired bound.

Hence, suppose $f(v_0)$ is an extreme value, say it is 0. (If instead it is the largest value, the span of f, just take the complementary labeling.) If f has more than $\frac{n}{2}$ labels which are $\geq 2k$, then the largest of them must be at least $2k + \frac{n}{2}$, which is the bound we want.

Else, there are at most $\frac{n}{2}$ labels which are $\geq 2k$. Then we have at least $\frac{n}{2}$ labels in $[k, 2k)$. Indeed, we have exactly this many labels in $[k, 2k)$ due to the fact that the corresponding vertices must be independent in W_n. Hence, we have exactly $\frac{n}{2}$ labels which are $\geq 2k$. The largest label in $[k, 2k)$ must be at least $k + \frac{n}{2} - 1$, while its two neighbors on the cycle in W_n must both have labels that are $\geq 2k + \frac{n}{2} - 1$; the larger one is $\geq 2k + \frac{n}{2}$. Hence the span of f is at least this value. $\square$ $\blacksquare$

This completes the proof of Theorem 2.3.

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