Real Number Channel Assignments for Lattices*

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Abstract

We consider the problem of assigning a numerical channel to each transmitter in a large regular array such that multiple levels of interference, which depend on the distance between transmitters, are avoided by sufficiently separating the channels. The goal is to find assignments that minimize the span of the labels used. A previous paper of the authors introduced a model for this problem using real number labellings of (possibly infinite) graphs G. Given reals $k_1, k_2, \ldots, k_p \geq 0$, one denotes by $\lambda(G; k_1, k_2, \cdots, k_p)$ the infimum of the spans of the labellings f of the vertices v of G, such that for any two vertices v and w, the difference in their labels is at least k_i, where i is the distance between v and w in G. When $p = 2$, it is enough to determine $\lambda(G; k, 1)$ for reals $k \geq 0$; for G of bounded maximum degree, this will be a continuous, piecewise linear function of k. Here we consider this function for infinite regular lattices that model large planar networks, building on earlier efforts by other researchers. For the triangular lattice, we determine the function for $k \geq 1$, which had previously been found for rational $k \geq 3$ by Calamoneri. We also give bounds for $0 \leq k \leq 1$. For the square lattice and the hexagonal lattice, we completely determine the function for $k \geq 0$, which had been given for rational $k \geq 3$ and $k \geq 2$, respectively, by Calamoneri.

Portions of it have been obtained by other researchers for infinite regular lattices that model large planar networks. Here we present the complete function $\lambda(G; k, 1)$, for $k \geq 1$ when G is the triangular, square, or hexagonal lattice.

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1 Introduction

Efficient channel assignment algorithms in wireless networks are increasingly important. There is usually a large network of transmitters in the plane, and a numerical channel must be assigned to each transmitter, where channels for nearby vertices must be assigned so as to avoid interference. The goal is to minimize the portion of the frequency spectrum that must be allocated to the problem, so it is desired to minimize the span of a feasible labelling.

Hale [22] (1980) formulated such channel assignment problems in network engineering as graph labelling problems: each transmitter is represented by a vertex, and any pair of vertices that may interfere is represented by an edge in the graph. All labels are integers.

In 1988 Lanfear proposed to Roberts [34] a new 2-level channel assignment problem of interest to NATO, in which integer labels are assigned to transmitters in the plane, with two levels of interference, depending on the distance between transmitters, say labels differ by at least two (respectively, one) when the transmitters are within some fixed distance A (resp., $2A$). Griggs introduced the analogous "lambda-labeling" problem for graphs, and made the initial investigation of this graph theory problem with Yeh [21]. They extended the problem in the natural way, by specifying separations $k_1, \ldots, k_p$ for vertices at distances $1, \ldots, p$, defining an $L(k_1, k_2, \cdots, k_p)$-labelling of a graph G to be an assignment of nonnegative numbers $f(v)$ to the vertices v of G, such that $|f(u) - f(v)| \geq k_i$ if u and v are at distance i in G. The labelling f is said to belong to the set $L(k_1, k_2, \cdots, k_p)(G)$. One denotes by $\lambda(G; k_1, k_2, \cdots, k_p)$ the minimum span over such f, where the span is the difference between the largest and smallest labels $f(v)$. Griggs and Yeh concentrated on the fundamental case of $L(2,1)$-labellings, and many authors have subsequently contributed to the literature on these $L(2,1)$ and more general labellings. For an online survey and annotated bibliography of work with conditions at distance $p = 2$, see [7]. For an overview of the recent progress and state of the general theory, refer to the recent survey [20].

In applications the frequency channel separations k_i for two transmitters are often taken to be inversely proportional to the physical distance i between them [3]. Most articles assume that the separations are nonincreasing, $k_1 \geq k_2 \geq \ldots \geq k_p$. But this is not required in the theory. Since in applications one can in principle use any frequencies (channels) in the available continuous frequency spectrum, not only from a discrete set, Griggs [17] proposed extending integer graph labellings to allow the labels and separations k_i to be nonnegative *real* numbers. They used the same notation as before, $L(k_1, \ldots, k_p)(G)$ and $\lambda(G; k_1, \ldots, k_p)$, but now the span of a real labelling is the difference between the supremum and the infimum of the labels used, and λ is the infimum of the spans of such labellings. The authors [19] first explored this concept for simple graphs, such as paths and cycles, and then began to study optimal labellings for the lattices considered in this paper. Their early results led to the discovery of properties for general graphs, which were included in the first, foundational paper [17]. The new insights and tools developed in that project are applied to lattices in this paper. The methods described here for lattices in turn are potentially applicable to other classes of graphs.

For graphs of bounded maximum degree, the authors proved the existence of an optimal labelling of a nice form, in which all labels belong to a discrete set, denoted by $D(k_1, k_2, \ldots, k_p)$, of linear combinations $\sum_i a_i k_i$, with nonnegative integer coefficients a_i.

Theorem 1.1 (The D-Set Theorem [17]). *Let G be a graph, possibly infinite, with finite maximum degree. Let p be a positive integer, and let k_i, $1 \leq i \leq p$ be real numbers ≥ 0. Then there exists a finite optimal $L(k_1, k_2, \ldots, k_p)$-labelling $f^* : V(G) \to [0, \infty)$ in which the smallest label is 0 and all labels belong to the set $D(k_1, k_2, \ldots, k_p)$. Hence, $\lambda(G; k_1, k_2, \ldots, k_p)$ belongs to $D(k_1, k_2, \ldots, k_p)$.*

Due to the D-set Theorem, previous optimal integer labelling results are compatible with new optimal real number labelling results. Some natural properties of distance-constrained labellings become more evident in the setting of real number labellings. In particular, the authors made the following observation, which was evident already in earlier work of Georges and Mauro on integer labellings.

Lemma 1.2 (Scaling). *For real numbers $d, k_i \geq 0$, $i = 1, 2, \ldots, p$,*

$$\lambda(G; d \cdot k_1, d \cdot k_2, \ldots, d \cdot k_p) = d \cdot \lambda(G; k_1, k_2, \ldots, k_p).$$

More subtle is the result of the authors [17, 24] that $\lambda(G; k_1, k_2, \ldots, k_p)$ is a continuous function of the separations k_i for any graph G (possibly infinite) with finite maximum degree. Hence, results about the minimum spans $\lambda(G; k_1, k_2, \ldots, k_p)$ for k_i being rational numbers can often be extended into the results for k_i being real numbers. Indeed, by the Scaling Lemma, it is usually enough to obtain results for integer k_i. The analysis is often more clear when considering real number labellings. For any fixed p and any graph G with finite maximum degree, the authors conjectured [17] that $\lambda(G; k_1, k_2, \ldots, k_p)$ is a piecewise linear function of real numbers k_i, where the pieces have nonnegative integer coefficients and where there are only finitely many pieces. The authors proved this if G is finite or if $p = 2$ [17]. Subsequently, Král' proved the full conjecture [26].

For the $p = 2$ case, the Scaling Lemma implies that for $k_2 > 0$, $\lambda(G, k_1, k_2) = k_2 \lambda(G; k, 1)$, where $k = k_1/k_2$. This reduces the two-parameter function to a one parameter function, $\lambda(G; k, 1)$, $k \geq 0$. As just discussed, it is a continuous, nondecreasing, piecewise linear function with finitely many pieces. Further, each piece has the form $ak+b$ for some nonnegative integers $a, b \geq 0$. This paper concerns the function $\lambda(G; k, 1), k \geq 0$, for the most natural infinite regular planar lattices (also called grid graphs), which are the triangular (6-regular), square (4-regular), and hexagonal (3-regular) lattices. The optimal span function is completely determined in the range of natural application, $k \geq 1$. It is solved as well for $0 \leq k \leq 1$ for the square and hexagonal lattice. For the triangular lattice, the problem appears to be much tougher for $0 \leq k \leq 1$, but here portions of it are solved and bounds are given in the remaining intervals. (See the last section for an update on this work.) Despite their nice properties, the optimal span function turns out to be surprisingly complicated for these three regular lattices.

Section 2 introduces some of the general methods used to obtain optimal lattice labellings. It also reviews some of the known results for labelling infinite trees with conditions at distance two, which are closely related to the lattice results. Sections 3,4 and 5 contain the results for the triangular, square, and hexagonal lattices, respectively. The detailed proofs, which make up most of the paper, are presented in Sections 5, 6, and 7, respectively. The paper concludes with ideas for future research directions.

2 Methods

Upper bounds on $\lambda(G; k_1, k_2, \ldots, k_p)$ are generally achieved by constructing an efficient labelling, sometimes discovered by computer search. Typically this can be achieved by coordinatizing the vertices of the lattice, giving an explicit labelling for a small piece, and repeating the pattern, tiling the whole lattice with congruent pieces. Lower bound proofs are generally more difficult. For these we identify crucial particular values of k where we need to prove a lower bound on $\lambda(G; k, 1)$. Such k are rational, say $k = a/b$ for some integers $a, b > 0$. By Scaling, it is then equivalent to bound $\lambda(G; a, b)$ below, which has the advantage that we need only consider integer $L(a, b)$-labellings, which have integer spans. We then seek to prove an integer bound, say $\lambda(G; a, b) \geq c$, by contradiction: If it is not true, then $\lambda(G; a, b) \leq c - 1$, and there must exist a labelling f of G using labels from the set $\{0, 1, \ldots, c - 1\}$. We restrict f to an appropriate finite induced subgraph of G, and argue that some label, call it L, must be avoided by f. We continue to eliminate possible labels, until there remains a set of labels for which it can be shown that in fact no feasible labelling exists. In some cases we had to write a computer program to check all possible labellings from a specified label set of a particular induced subgraph.

We begin by recording a simple way to expand the set of avoided labels by using symmetry (which was introduced for the triangular lattice in [4, 14]):

Lemma 2.1 (Symmetry)**.** *Let S, L, and $k_1, k_2, \ldots, k_p$ be nonnegative integers, and let G be a graph. If every $L(k_1, k_2, \ldots, k_p)(G)$-labelling f into $\{0, \ldots, S\}$ avoids (respectively, uses) label L, then every such labelling f avoids (respectively, uses) label $S - L$.*

We next describe a simple method for general graphs G that is surprisingly useful, for it permits us to extend a bound at some particular value to general values of k:

Lemma 2.2. *Let a, b be reals with $a > 0$.*
If $\lambda(G; a, 1) \leq b$, then $\lambda(G; k, 1) \leq \begin{cases} b & \text{if } 0 \leq k \leq a \\ \frac{b}{a}k & \text{if } k \geq a \end{cases}$.
If $\lambda(G; a, 1) \geq b$, then $\lambda(G; k, 1) \geq \begin{cases} \frac{b}{a}k & \text{if } 0 \leq k \leq a \\ b & \text{if } k \geq a \end{cases}$.
In particular, if $\lambda(G; a, 1) = b$, then
For $0 \leq k \leq a$, $\frac{b}{a}k \leq \lambda(G; k, 1) \leq b$;
For $k \geq a$, $b \leq \lambda(G; k, 1) \leq \frac{b}{a}k$.

Proof: If $\lambda(G; a, 1) \leq b$, we have:

- For $0 \le k \le a$, the result follows from the fact that $\lambda(G; k, 1)$ is nondecreasing.
- For $k \ge a$, we also use the Scaling Lemma to obtain $\lambda(G; k, 1) \le \lambda(G; k, \frac{k}{a}) = \frac{k}{a}\lambda(G; a, 1) \le \frac{b}{a}k$.

The proof is similar, if $\lambda(G; a, 1) \ge b$. ■

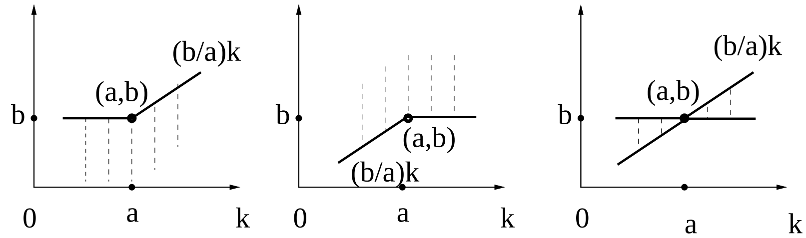


Figure 1: The bound on $\lambda(G; k, 1)$

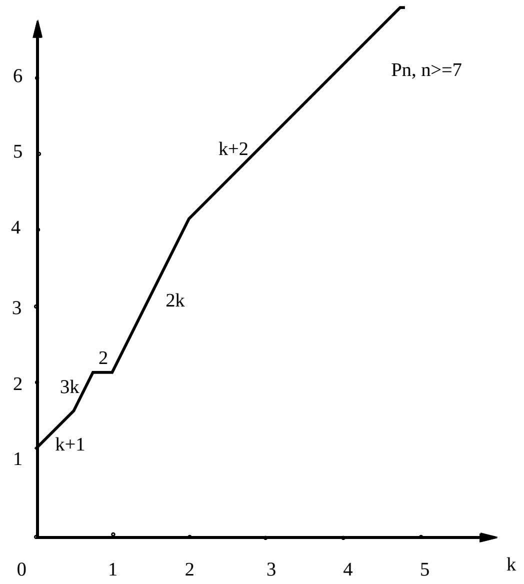


Figure 2: The minimum span $\lambda(P_n; k, 1)$ for path $P_n, n \ge 7$.

It is interesting to compare the lattice problems to those for infinite trees. For integer $d > 0$, let T_d denote the tree that is regular of degree d. Note that T_d is infinite for $d \ge 2$ and T_2 is an infinite path. For the path P_n on n vertices, $n \ge 7$, the authors [19] determined the minimum span $\lambda(P_n; k, 1), n \ge 7$ (see Figure 2).

Georges and Mauro [12] obtained the values of $\lambda(T_d; k_1, k_2)$ for integers $k_1 \ge k_2 \ge 0$. In a subsequent paper (with the same title!) Calamoneri, Pelc and Petreschi [8] gave the values for integers $0 \le k_1 \le k_2$. By continuity and scaling, these can be restated in terms of $\lambda(T_d; k, 1)$ for reals $k \geq 0$, which is neater, so we use this format here. For $k \geq 1$ the functions get increasingly complicated as d grows, so we only state formulas for the values required here, $d = 3, 4$:

Theorem 2.3 ([12]). *For real $k \geq 1$ we have*

$$\lambda(T_3; k, 1) = \begin{cases} 3k & \text{if } 1 \leq k \leq \frac{3}{2} \\ k+3 & \text{if } \frac{3}{2} < k \leq 2 \\ 2k+1 & \text{if } 2 \leq k \leq 3 \\ k+4 & \text{if } k \geq 3 \end{cases}$$

Theorem 2.4 ([12]). *For real $k \geq 1$, we have*

$$\lambda(T_4; k, 1) = \begin{cases} 4k & \text{if } 1 \leq k \leq \frac{4}{3} \\ k+4 & \text{if } \frac{4}{3} < k \leq \frac{3}{2} \\ 3k+1 & \text{if } \frac{3}{2} \leq k \leq \frac{5}{3} \\ 6 & \text{if } \frac{5}{3} \leq k \leq 2 \\ 3k & \text{if } 2 \leq k \leq \frac{5}{2} \\ k+5 & \text{if } \frac{5}{2} \leq k \leq 3 \\ 2k+2 & \text{if } 3 \leq k \leq 4 \\ k+6 & \text{if } k \geq 4 \end{cases}$$

Theorem 2.5 ([8]). *For real k, $0 \leq k \leq 1$, and integer $d \geq 2$, we have*

$$\lambda(T_d; k, 1) = \begin{cases} k + (d-1) & \text{if } 0 \leq k \leq \frac{1}{2} \\ (2d-1)k & \text{if } \frac{1}{2} < k \leq \frac{d}{2d-1} \\ d & \text{if } \frac{d}{2d-1} \leq k \leq 1 \end{cases}$$

Next we give two results that relate the optimal spans of regular trees T_d to that of general d-regular graphs G.

Theorem 2.6 ([13]). *Let G be a regular graph of degree $d \geq 2$. Then for all real $k \geq 1$, we have $\lambda(G; k, 1) \geq \lambda(T_d; k, 1)$.*

The idea of the proof is that T_d is a universal cover of G, so that we can easily define a graph homomorphism h from T_d to G. Then if f is an optimal $L(k, 1)$-labelling of G, $f \circ h$ is an $L(k, 1)$-labelling of T_d, so that

$$\lambda(T_d; k, 1) \leq \text{span}(f) \leq \text{span}(f') = \lambda(G; k, 1).$$

The condition $k \geq 1$ above is certainly necessary, since it could be for vertices s and t at distance two that $h(s)$ and $h(t)$ are adjacent, and we would only be certain that $|f(s) - f(t)| \geq k$, which is not strong enough, if $k < 1$. For instance, let $k < 1$. If $d = 2$, then T_d is an infinite path, and we may consider the 2-regular graph $G = C_3$. It is easily seen (by examining the two neighbors of a vertex with label 0) that $\lambda(T_2; k, 1) \geq 1 + k$, which exceeds $\lambda(C_3; k, 1) = 2k$.

However, if G is triangle-free, then it cannot be that $h(s)$ and $h(t)$ are adjacent in the problematic case above.

Theorem 2.7. *Let G be a triangle-free regular graph of degree $d \geq 2$. Then for all real $k \geq 0$, we have $\lambda(G; k, 1) \geq \lambda(T_d; k, 1)$.* ■

3 The Triangular Lattice

In a radio mobile network, the large service areas are often covered by a network of nearly congruent polygonal cells, with each transmitter at the center of a cell that it covers. A honeycomb of hexagonal cells provides the most economic covering of the whole plane [11] (i.e., covers the plane with smallest possible transmitter density), where the transmitters are placed in the triangular lattice Γ_Δ (see Figure 3). We fix a point to be the original point o and impose an xoy coordinate system so that we can name each point by its xoy coordinate.

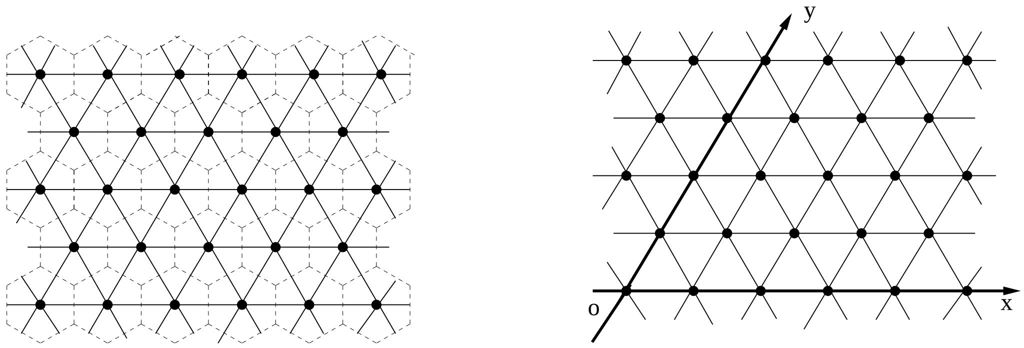


Figure 3: The Hexagonal Cell Covering and the Triangular Lattice Γ_Δ

This problem has some history, owing to the fundamental nature of the triangular lattice for channel assignment problems. Griggs [15] formulated an integer $L(k, 1)$-labelling problem on the triangular lattice Γ_Δ for the 2000 International Math Contest in Modeling (MCM). Among 271 teams which worked on this problem for four days and wrote papers, five teams [4, 10, 14, 30, 35] won the contest and got their papers published. As pointed out in the last section, two student teams observed what became the Symmetry Lemma 2.1. All winners found $\lambda(\Gamma_\Delta; k, 1)$ for $k = 2, 3$, and some gave labellings for $k = 1$ or for integers $k \geq 4$ that turn out to be optimal, but without proving the lower bound. The team of Goodwin, Johnston and Marcus [14] proved the optimality for integers $k \geq 4$ (quite an achievement in such a short time) and considered the more general problem of $\lambda(\Gamma_\Delta; k_1, k_2)$ for integers k_1, k_2. Subsequently, researchers Yeh [25] and Zhu and Shi [36] took over, each solving some special cases for integers $k_1 \geq k_2$. Calamoneri [6] gave the minimum span for integers $k_1 \geq 3k_2$, and she gave bounds for $k_2 \leq k_1 \leq 3k_2$.

Here we describe the solution of the $L(k, 1)$-labelling problem for the triangular lattice for real numbers $k \geq 1$, and we give bounds for $0 \leq k \leq 1$ (see Figure 3), where considerable effort has not yet led to a full solution.

Theorem 3.1. *For $k \geq 0$ the minimum span of any $L(k,1)$-labelling of the triangular lattice is given by:*

$$\lambda(\Gamma_\Delta; k, 1) \begin{cases} = 2k+3 & \text{if } 0 \leq k \leq \frac{1}{3} \\ \in [2k+3, 11k] & \text{if } \frac{1}{3} \leq k \leq \frac{9}{22} \\ \in [2k+3, \frac{9}{2}] & \text{if } \frac{9}{22} \leq k \leq \frac{3}{7} \\ \in [9k, \frac{9}{2}] & \text{if } \frac{3}{7} \leq k \leq \frac{1}{2} \\ \in [\frac{9}{2}, \frac{16}{3}] & \text{if } \frac{1}{2} \leq k \leq \frac{2}{3} \\ \in [\frac{16}{3}, \frac{23}{4}] & \text{if } \frac{2}{3} \leq k \leq \frac{3}{4} \\ \in [\frac{23}{4}, 6] & \text{if } \frac{3}{4} \leq k \leq \frac{4}{5} \\ = 6 & \text{if } \frac{4}{5} \leq k \leq 1 \\ = 6k & \text{if } 1 \leq k \leq \frac{4}{3} \\ = 8 & \text{if } \frac{4}{3} \leq k \leq 2 \\ = 4k & \text{if } 2 \leq k \leq \frac{11}{4} \\ = 11 & \text{if } \frac{11}{4} \leq k \leq 3 \\ = 3k+2 & \text{if } 3 \leq k \leq 4 \\ = 2k+6 & \text{if } k \geq 4 \end{cases}.$$

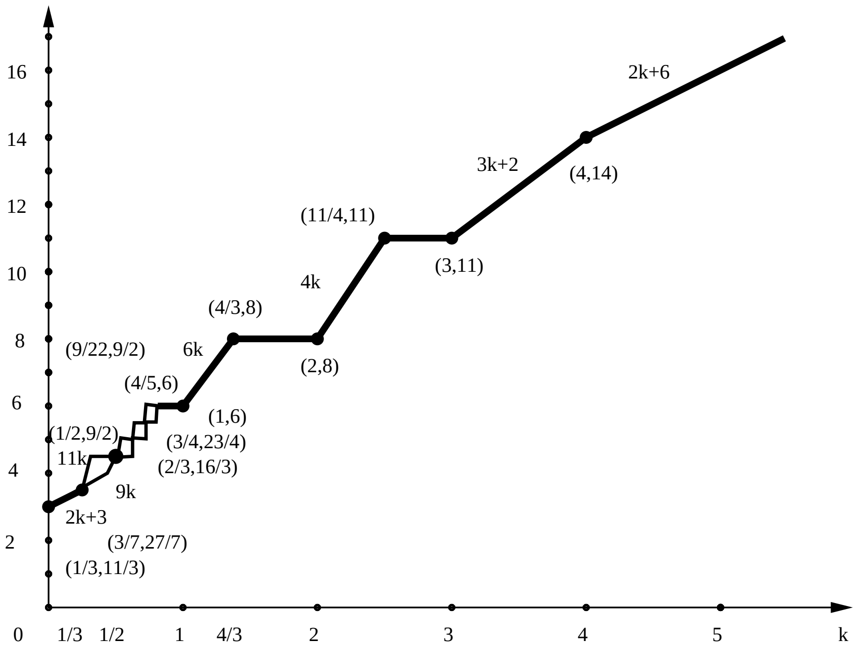


Figure 4: $\lambda(\Gamma_\Delta; k, 1)$ for $k \geq 0$.

For the proof of this theorem, go to Section 6. We can use Lemma 2.2 to give a slight improvement to the stated bounds in the interval that is not yet resolved, $1/3 \leq k \leq 4/5$: having the exact values of lambda at $k = 2/3$, $3/4$, $4/5$ means that there is a linear lower bound for k just below these values, of $8k$, if $k \in [\frac{9}{16}, \frac{2}{3}]$; of $\frac{23k}{3}$, if $k \in [\frac{16}{23}, \frac{3}{4}]$; and of $\frac{15k}{2}$, if $k \in [\frac{23}{30}, \frac{4}{5}]$. Similarly, there is a linear upper bound for k just above these values, of $9k$, if $k \in [\frac{1}{2}, \frac{16}{27}]$; of $8k$, if $k \in [\frac{2}{3}, \frac{23}{32}]$; and of $\frac{23k}{3}$, if $k \in [\frac{3}{4}, \frac{18}{23}]$. We conjecture that the upper bound on $\lambda(\Gamma_\Delta; k, 1)$ is the actual value for $\frac{1}{3} \leq k \leq \frac{1}{2}$. For $\frac{1}{2} \leq k \leq \frac{4}{5}$, we conjecture that $\lambda(\Gamma_\Delta; k, 1) = 5k + 2$, a formula which works already in this interval at $k = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ and $\frac{4}{5}$. Incidentally, we compared the formulas for the triangular lattice (which is 6-regular) to that of the regular infinite tree, T_6, and found they are quite different, not worth stating explicitly here.

4 The Square Lattice

Inside cities the high buildings can be obstacles in the signal path and limit the range of a cell. A Manhattan cellular system [3] can be used that is modeled by the square lattice $\Gamma_\square$ (see Figure 4). Many graphs corresponding to cellular systems are the induced subgraphs of the square lattice and the triangular lattice.

Theorem 4.1 is the complete determination of $\lambda(\Gamma_\square; k, 1)$ for real numbers $k \geq 0$ (see Figure 6). Previously, Calamoneri [6] independently gave the minimum (integer) span $\lambda(\Gamma_\square; k_1, k_2)$ for integers $k_1 \geq 3k_2$, as well as bounds when $k_2 \leq k_1 \leq 3k_2$. (It should be noted that the stated bounds in the earlier extended abstract [5] are not entirely correct, such as the claim that $\lambda(\Gamma_\square; 3, 2) = 12$, which is contradicted by the $L(3,2)$-labelling of span only 11 from [23]. However, the bounds in the subsequent preprint [6] appear to be correct.)

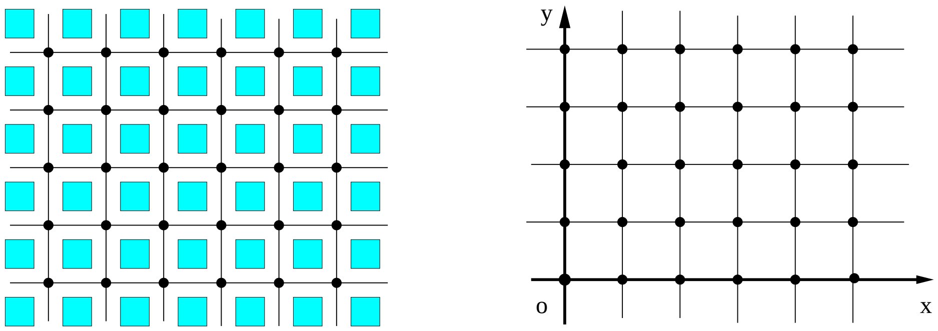


Figure 5: A Manhattan Network and the Square Lattice $\Gamma_\square$

Theorem 4.1. *For $k \geq 0$ the minimum span of any $L(k,1)$-labelling of the square lattice is given by:*

$$\lambda(\Gamma_{\square}; k, 1) \;=\; \begin{cases} k+3 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 7k & \text{if } \frac{1}{2} < k \leq \frac{4}{7} \\ 4 & \text{if } \frac{4}{7} \leq k < 1 \\ 4k & \text{if } 1 \leq k \leq \frac{4}{3} \\ k+4 & \text{if } \frac{4}{3} < k \leq \frac{3}{2} \\ 3k+1 & \text{if } \frac{3}{2} < k \leq \frac{5}{3} \\ 6 & \text{if } \frac{5}{3} \leq k \leq 2 \\ 3k & \text{if } 2 < k \leq \frac{8}{3} \\ 8 & \text{if } \frac{8}{3} \leq k \leq 3 \\ 2k+2 & \text{if } 3 \leq k \leq 4 \\ k+6 & \text{if } k \geq 4 \end{cases} .$$

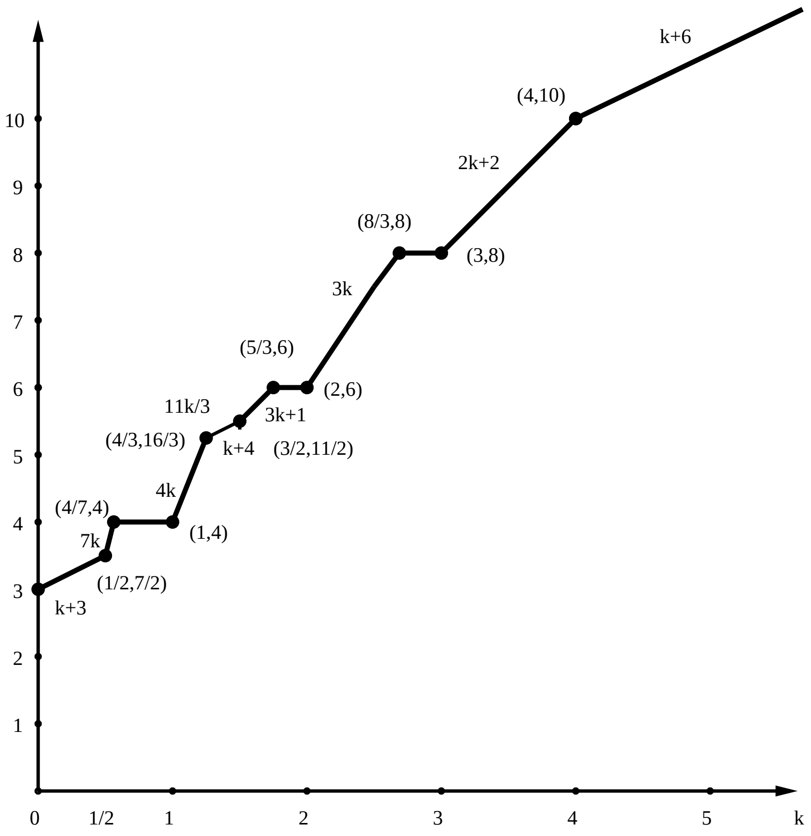


Figure 6: The Minimum Span $\lambda(\Gamma_{\square}; k, 1)$

This Theorem, which is proven in Section 7, allows us to answer a question posed by Georges and Mauro (private communication): does $\lambda(\Gamma_{\square}; k, 1)$ agree with $\lambda(T_4; k, 1)$ for all $k \geq 0$, as stated in Theorems 2.4 and 2.5? Since $\Gamma_{\square}$ is a triangle-free regular graph of degree 4, Theorem 2.7 is applicable, and tells us that $\lambda(\Gamma_{\square}; k, 1) \geq \lambda(T_4; k, 1)$ for all

$k \geq 0$. Indeed, they almost always agree. However, there is one interval in which they differ: when $\frac{5}{2} < k < 3$, $\lambda(\Gamma_\Box; k, 1) = \min\{3k, 8\}$ is strictly larger than $\lambda(T_4; k, 1) = k+5$.

5 The Hexagonal Lattice

Another interesting fundamental planar array is the hexagonal lattice Γ_H (see Figure 7), which is the dual of the triangular lattice. We are not aware of its being used in real life for wireless networks, but it is mentioned in the engineering literature.

We designate a point o to be the origin, and we impose an xoy coordinate system so that we can name each point by its xoy coordinate, where (i, j) are vertices (see Figure 7). Vertices (i, j) and $(i+1, j)$ are always adjacent, while vertices (i, j) and $(i, j+1)$ are adjacent if and only if $i \equiv j (\mod 2)$. Calamoneri [6] gives the minimum span for the hexagonal lattice for integers $k_1 \geq 2k_2$ and bounds for $k_2 \leq k_1 \leq 2k_2$. Here we resolve the remaining open cases and completely determine the span $\Gamma_H; k, 1)$ for real numbers $k \geq 0$ (see Figure 5).

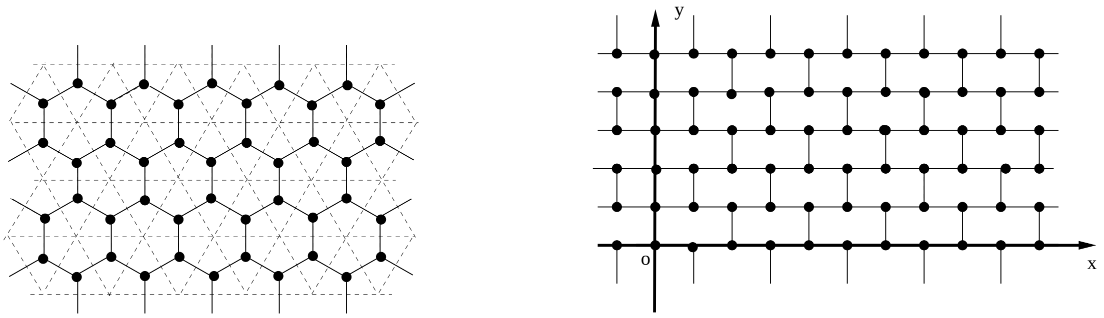


Figure 7: The Equilateral Triangle Cell Covering and the Hexagonal Lattice Γ_H

Theorem 5.1. *For $k \geq 0$ the minimum span of any $L(k, 1)$-labelling of the hexagonal lattice is given by:*

$$\lambda(\Gamma_H; k, 1) = \begin{cases} k+2 & \text{if } 0 \leq k \leq \frac{1}{2} \\ 5k & \text{if } \frac{1}{2} \leq k \leq \frac{3}{5} \\ 3 & \text{if } \frac{3}{5} \leq k \leq 1 \\ 3k & \text{if } 1 \leq k \leq \frac{5}{3} \\ 5 & \text{if } \frac{5}{3} \leq k \leq 2 \\ 2k+1 & \text{if } 2 \leq k \leq 3 \\ k+4 & \text{if } k \geq 3 \end{cases}.$$

The proof is given in Section 8. We may now compare the spans of the hexagonal lattice and the regular tree of the same degree, T_3. As before, the fact that Γ_H is triangle-free allows us to apply Theorem 2.7 to see that $\lambda(\Gamma_H; k, 1) \geq \lambda(T_3; k, 1)$ for all $k \geq 0$. Comparing the formula above for Γ_H to those from Theorems 2.3 and 2.5, we see that

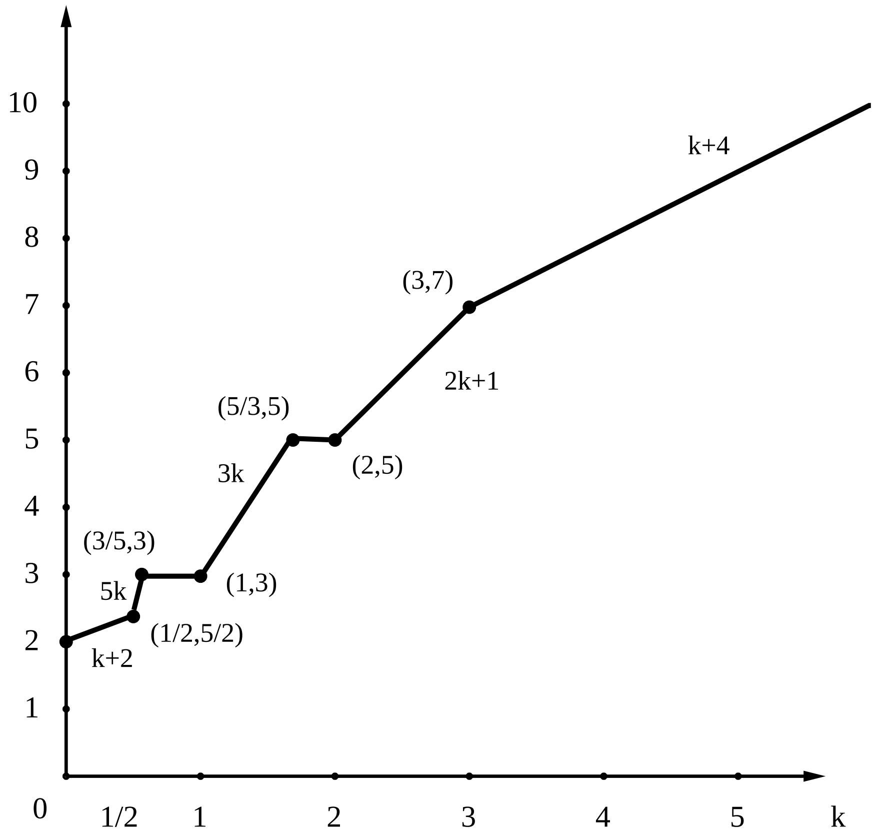


Figure 8: The Minimum Span $\lambda(\Gamma_H; k, 1)$ for $k \geq 0$.

Γ_H agrees with T_3 except in the range $\frac{3}{2} < k < 2$, where $\Gamma_H; k, 1) = \min\{3k, 5\}$ is strictly larger than $\lambda(T_3; k, 1) = k + 3$.

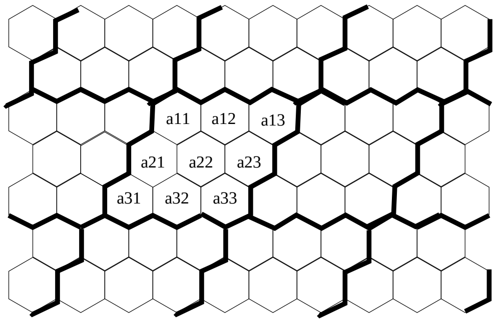
6 The Proof of the Triangular Lattice Theorem 3.1

Generally, we get upper bounds by constructing feasible labellings. For lower bounds we derive contradictions on induced subgraphs for labellings of smaller span. Especially, we denote by B_7 (resp., B_{19}, B_{37}) the induced subgraph of Γ_Δ on all 7 (resp., 19, 37) vertices at distance at most one (resp., two, three) from a fixed vertex. Lemma 2.2 is particularly useful for obtaining lower and upper bounds.

To find an upper bound on $\lambda(\Gamma_\Delta; k, 1)$, one construction method is to tile the whole lattice by a labelled parallelogram described by a matrix of labels. We define a doubly periodic labelling of the triangular lattice by an $m \times n$ labelling matrix $A := [a_{i,j}]$, such that we label point (i, j) by $a_{m-(j \bmod m),\ (i \bmod n)+1}$, where i, j are integers.For example, the following labelling (see Figure 9) is defined by the labelling matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then Figure 9 shows how the labels are assigned, where $a_{3,1}$ is at the vertex with coordinates $(0,0)$ in the triangular lattice. The whole lattice is tiled with copies of the 3×3 tile as shown.



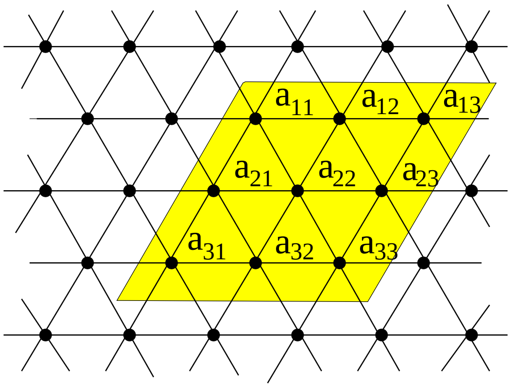


Figure 9: The Matrix Labelling

A special case of matrix labelling is defined simply by "arithmetic progressions": For positive integers k_1, k_2, we construct a labelling $f \in L(k_1, k_2)$ by taking $f(i,j) = (ai + bj) \bmod l$, for positive integers a, b, l, When such f is feasible, we obtain $\lambda(\Gamma_\Delta; k_1, k_2) \leq l - 1$. Some labellings of this kind were given for the triangular and square lattices in [23]. We found some new arithmetic progression labellings by computer search. We begin our constructions at $k = 0$:

Proposition 6.1. *For $0 \leq k \leq \frac{1}{3}$, we have $\lambda(\Gamma_\Delta; k, 1) \leq 2k + 3$. For $\frac{1}{3} \leq k \leq \frac{9}{22}$, we have $\lambda(\Gamma_\Delta; k, 1) \leq 11k$.*

Proof: We get the upper bound $\lambda(\Gamma_\Delta; k, 1) \leq 2k + 3$ for $0 \leq k \leq \frac{1}{3}$ by defining the labelling matrix

$$A = \begin{bmatrix} k+1 & 2k & 0 & k+3 & 2k+2 & 2 \\ 2k+1 & 1 & k & 2k+3 & 3 & k+2 \\ 0 & k+3 & 2k+2 & 2 & k+1 & 2k \\ k & 2k+3 & 3 & k+2 & 2k+1 & 1 \\ 2k+2 & 2 & k+1 & 2k & 0 & k+3 \\ 3 & k+2 & 2k+1 & 1 & k & 2k+3 \end{bmatrix}.$$

In particular, $\lambda(\Gamma_\Delta; \frac{1}{3}, 1) \leq \frac{11}{3}$, and Lemma 2.2 implies that $\lambda(\Gamma_\Delta; k, 1) \leq 11k$ for $k \geq \frac{1}{3}$. ■

Next, we can improve upon the $11k$ upper bound for k between $9/22$ and $1/2$:

Proposition 6.2. *For* $\frac{9}{22} \leq k \leq \frac{1}{2}$*, we have* $\lambda(\Gamma_\Delta; k, 1) \leq \frac{9}{2}$.

Proof: The upper bound $\lambda(1,2) \leq 9$ is given in [25] by an arithmetic progression labelling in $L(1,2)$: Label point (i,j) by $(i+4j) \bmod 10$. By scaling, this gives the bound on $\lambda(\Gamma_\Delta; \frac{1}{2}, 1)$, which then extends to $k \leq \frac{1}{2}$ by Lemma 2.2. ■

The upper bound for k between $\frac{1}{2}$ and $\frac{3}{4}$ follows from the bounds at $k = \frac{2}{3}$ and $\frac{3}{4}$ by the fact that $\lambda(\Gamma_\Delta; k, 1)$ is nondecreasing (Lemma 2.2):

Proposition 6.3. *1. We have* $\lambda(\Gamma_\Delta; 2, 3) \leq 16$*. Hence,* $\lambda(\Gamma_\Delta; \frac{2}{3}, 1) \leq \frac{16}{3}$.

2. We have $\lambda(\Gamma_\Delta; 3, 4) \leq 23$*. Hence,* $\lambda(\Gamma_\Delta; \frac{3}{4}, 1) \leq \frac{23}{4}$.

Proof: By computer search of arithmetic progression labellings, we discovered $f_1 \in L(2,3)(\Gamma_\Delta)$ given by $f_1(i,j) = (2i+7j) \bmod 17$ and $f_2 \in L(3,4)(\Gamma_\Delta)$ given by $f_2(i,j) = (3i+10j) \bmod 24$. Hence, $\lambda(\Gamma_\Delta; 2, 3) \leq 16$ and $\lambda(\Gamma_\Delta; 3, 4) \leq 23$. ■

Next we extend the upper bound out to $k = \frac{4}{3}$ by applying Lemma 2.2 with the upper bound on $\lambda(\Gamma_\Delta; 1, 1)$. Note that the upper bounds we are giving here for $k = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and 1 are matched by the lower bounds, so give the correct values of $\lambda(\Gamma_\Delta; k, 1)$ for these k.

Proposition 6.4 ([4, 14])**.** *We have* $\lambda(\Gamma_\Delta; 1, 1) = 6$.
Hence, $\lambda(G; k, 1) \leq \begin{cases} 6 & \text{if } \frac{3}{4} \leq k \leq 1 \\ 6k & \text{if } 1 \leq k \leq \frac{4}{3} \end{cases}$.

Proof: We get the upper bound, $\lambda(B_7; 1, 1) \leq 6$, from the arithmetic progression labelling $f(i,j) = (i+3j) \bmod 7$. The rest follows from Lemma 2.2. ■

We now use the construction of numerous MCM teams at $k = 2$ to extend our upper bound out to $k = \frac{11}{4}$, and another construction out to $k = 4$:

Proposition 6.5 ([4, 10, 14, 30, 35])**.** *We have* $\lambda(\Gamma_\Delta; 2, 1) \leq 8$.
Hence, $\lambda(G; k, 1) \leq \begin{cases} 8 & \text{if } \frac{4}{3} \leq k \leq 2 \\ 4k & \text{if } 2 \leq k \leq \frac{11}{4} \end{cases}$.

Proof: Label point (i,j) by $(2i+5j) \bmod 9$. ■

Proposition 6.6 ([14, 30])**.** *For* $3 \leq k \leq 4$*, we have* $\lambda(\Gamma_\Delta; k, 1) \leq 3k+2$.

Proof: We get the bound by defining the labelling matrix

$$A = \begin{bmatrix} 3k & 0 & k & 2k \\ 1 & k+1 & 2k+1 & 3k+1 \\ k+2 & 2k+2 & 3k+2 & 2 \end{bmatrix}. \quad ■$$

Using $\lambda(\Gamma_\Delta; 3, 1) \leq 11$, this bound of 11 extends down to $k = \frac{11}{4}$ by Lemma 2.2 A construction from the winning MCM papers takes care of all large k:

Proposition 6.7 ([4, 10, 14, 30, 35])**.** *For* $k \geq 4$*, we have* $\lambda(\Gamma_\Delta; k, 1) \leq 2k+6$.

Proof We get the labelling from the matrix

$$A = \begin{bmatrix} 2k+5 & 0 & k+4 \\ 1 & k+2 & 2k+6 \\ k+3 & 2k+4 & 2 \end{bmatrix}. \quad \blacksquare$$

We verify the lower bounds using proofs by contradiction (which can be rather complicated) and Lemma 2.2. We shall postpone the small values, $k \leq \frac{3}{4}$. We demonstrate two main methods of proof. The first method, for integers k_1, k_2, involves the successive elimination of possible labels, until a contradiction is reached. This method was used in the contest paper of Goodwin *et al.* to handle the case of integers $k \geq 4$ (see our comments before Proposition 6.11). We also drew ideas from [36] for the proof of the following important case.

Proposition 6.8. *We have* $\lambda(\Gamma_\Delta; 4, 3) \geq 24$.
Hence, $\lambda(\Gamma_\Delta; k, 1) = \begin{cases} 6k & \text{if } \frac{3}{4} \leq k \leq \frac{4}{3} \\ 8 & \text{if } \frac{4}{3} \leq k \leq 2 \end{cases}$

Proof: The first statement implies the second by Lemma 2.2. It suffices to prove that $\lambda(\Gamma_\Delta; 4, 3) \geq 24$. Assume to the contrary that there exists a labelling $f \in L(4,3)(\Gamma_\Delta)$ with its labels in $\{0, 1, \ldots, 23\}$. The series of claims that follows restricts the labels f one can use until we find that no such f can exist at all, proving the proposition.
Claim 1. *The labelling* f *cannot use label* 3 *or* 20.
Proof: Assume f uses label 3 at v. By the separation conditions, the six labels around v belong to $\{7, 8, \ldots, 23\}$, and the difference between any pair of them is at least 3.

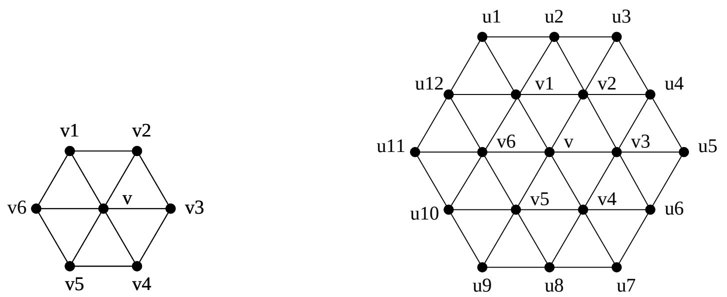


Figure 10: The Subgraphs B_7 and B_{19} of the Triangular Lattice.

Among all 49 possible labellings of B_7 with central label 0 by symmetry, we found by computer that there are just five feasible labellings of subgraph B_{19} that use 3 at the center (B_7, B_{19} are shown in Figure 10), and none of these can be extended to B_{37}. Full details are in [24]. By the Symmetry Lemma 2.1, f is also excluded from using the complementary label $23 - 3 = 20$, and the Claim follows.
Claim 2. *The labelling* f *cannot use label* 7 *or* 16.
Proof: Assume f uses label 7 at $v \in V(\Gamma_\Delta)$. Denote the six labels around v by $x_1 < x_2 \cdots < x_6$. By the separation conditions, $x_{i+1} \geq x_i + 3$ for $i = 1, 2, \ldots, 5$, and each

$x_i \in \{0,1,2,11,12,\ldots,19,21,22,23\}$ (recall we cannot use 3 or 20). Then, even if $x_1 \leq 2$, we must have $x_2 \geq 11$, $x_3 \geq 14$, $x_4 \geq 17$, $x_5 \geq 21$, $x_6 \geq 24$, a contradiction proving the Claim.

Now f has no label $3,7,16,20$. The proofs of Claims 3, 4, and 5 are similar to the proof of Claim 2, so we omit the details.
Claim 3. *The labelling f cannot use label* 6 *or 17.*
Claim 4. *The labelling f cannot use label* 10 *or* 13.
Claim 5. *The labelling f cannot use label* 11 *or* 12.

Now the set of all possible labels is $\{0,1,2,4,5,8,9,14,15,18,19,21,22,23\}$. We cannot find seven distinct labels, such that the difference between any two of them is at least 3. So we cannot label B_7, which is a contradiction. Thus, $\lambda(\Gamma_\Delta;4,3) \geq 24$. ■

By similar proofs, we have the following bounds (see [24] for full details).

Proposition 6.9. *1. We have $\lambda(\Gamma_\Delta;11,4) \geq 44$.*
Hence, $\lambda(\Gamma_\Delta;k,1) \geq \begin{cases} 4k & \text{if } 2 \leq k \leq \frac{11}{4} \\ 11 & \text{if } \frac{11}{4} \leq k \leq 3 \end{cases}$.

2. We have $\lambda(\Gamma_\Delta;1,2) \geq 9$.
Hence, $\lambda(\Gamma_\Delta;k,1) \geq \begin{cases} 9k & \text{if } \frac{3}{7} \leq k \leq \frac{1}{2} \\ \frac{9}{2} & \text{if } k \geq \frac{1}{2} \end{cases}$

3. We have $\lambda(\Gamma_\Delta;2,3) \geq 16$. Hence, $\lambda(\Gamma_\Delta;k,1) \geq \frac{16}{3}$ for $k \geq \frac{2}{3}$.

4. We have $\lambda(\Gamma_\Delta;3,4) \geq 23$. Hence, $\lambda(\Gamma_\Delta;x,1) \geq \frac{21}{4}$ for $k \geq \frac{3}{4}$.

5. We have $\lambda(\Gamma_\Delta;4,5) \geq 30$. Hence, $\lambda(\Gamma_\Delta;x,1) \geq 6$ for $k \geq \frac{4}{5}$.

The next result takes care of all k in the interval $(3,4)$. It can be derived by continuity and scaling from the corresponding result by Calamoneri [6] for integer labellings that give $\lambda(\Gamma_\Delta;k_1,k_2)$ for integers k_1,k_2 with $3k_2 \leq k_1 \leq 4k_2$. Her lower bound method involves looking at a small induced subgraph of the lattice and checking cases according to the numerical order of the labels. This is similar to the method devised independently by Georges and Mauro for labelling trees [12]. We discovered the result independently (but waited on the rest of this project before writing it up here). Because our proof illustrates a different method with some potential for future value, we include it here. It involves the successive removal of intervals of possible labels until there is a contradiction.

Proposition 6.10. *For $3 < k < 4$, we have $\lambda(\Gamma_\Delta;k,1) = 3k+2$.*

Proof: The upper bound comes from Proposition 6.6. We prove the lower bound by contradiction: Assume $\lambda(\Gamma_\Delta;k,1) = l < 3k+2$. By the D-Set Theorem, there is an optimal labelling $f \in L(k,1)(\Gamma_\Delta)$ with $\text{span}(f) = l < 3k+2$ and $f(u) = 0$ for some $u \in V(\Gamma_\Delta)$.
Claim 1. *The labelling f cannot use labels in $[k-1,k) \cup (l-k,l-k+1]$.*
Proof: Assume $f(v) \in [k-1,k)$ for some $v \in V(\Gamma_\Delta)$. The neighbors of v induce a C_6 subgraph, and their labels are all $\geq f(v)+k$. Hence, $\lambda(\Gamma_\Delta;k,1) \geq f(v)+k+\lambda(C_6;k,1) \geq (k-1)+k+(k+3) = 3k+2$ (because $\lambda(C_6;k,1) = k+3$ for $k \geq 3$, see [24]). It gives

a contradiction. Thus $f(v) \notin [k-1,k)$ for all $v \in V(\Gamma_\Delta)$. By symmetry of the labels, $f(v) \notin (l-k, l-k+1]$ for all $v \in V(\Gamma_\Delta)$, which proves the claim.

Now, $f(v) \in I_1 \cup I_2 \cup I_3$ for all $v \in V(\Gamma_\Delta)$, where $I_1 = [0, k-1), I_2 = [k, l-k], I_3 = (l-k+1, l]$. Then $|I_1| = k-1 < k, |I_2| = l-2k < k+2, |I_3| = k-1 < k$.

Claim 2. *The labelling f cannot use labels in $[k, k+1) \cup (l-k-1, l-k]$.*

Proof: Assume $f(v) \in [k, k+1)$ for some $v \in V(\Gamma_\Delta)$. Among the six distinct labels around v, at most one label is in $I_1 = [0, k-1)$ (because this label is $\leq f(v) - k < 1$), at most two labels are in $I_2 = [k, l-k]$ (because these two labels are $\geq f(v) + k \geq 2k$ and $|[2k, l-k]| = l - 3k < 2$), and at most three labels are in $I_3 = (l-k+1, l]$ (because $|I_3| < k$, these labels cannot be adjacent). Thus, one label is in I_1, two labels are in I_2, and three labels are in I_3. The three labels in I_3 are for vertices that aren't adjacent. The smallest of the three labels then must be next to at least one of the labels in I_2. This smallest label in I_3 is $\leq l-2 < 3k$. But the two labels in I_2 are $\geq f(v) + k \geq 2k$, and the smallest label in I_3, being next to one of these, must then be at least $3k$, a contradiction.

Thus, $f(v) \notin [k, k+1)$ for all $v \in V(\Gamma_\Delta)$. By symmetry of the labels, $f(v) \notin (l-k-1, l-k]$, proving the claim.

Now, $f(v) \in I_1 \cup I_2' \cup I_3 = [0, k-1) \cup [k+1, l-k-1] \cup (l-k+1, l]$, where $I_2' = [k+1, l-k-1]$. Then $|I_2'| \leq l - 2k - 2 < k$.

Claim 3. *The labelling f cannot use labels in $[k+1, k+2) \cup (l-k-2, l-k-1]$.*

Proof: Assume $f(v) \in [k+1, k+2)$ for some $v \in V(\Gamma_\Delta)$. Among the six labels around v, at most two labels are in $I_1 = [0, k-1)$ (because these two labels are $\leq f(v) - k < 2$), no label is in $I_2' = [k+1, l-k-1]$ (because if it exists, it would be $\geq f(v) + k \geq 2k+1 > l-k-1$, a contradiction), and at most three labels are in $I_3 = (l-2, l]$. We cannot label all six vertices.

Thus $f(v) \notin [k+1, k+2)$ for all $v \in V(\Gamma_\Delta)$. By symmetry of the labels, $f(v) \notin (l-k-2, l-k-1]$, proving the claim.

Now, $f(v) \in I_1 \cup I_2'' \cup I_3 = [0, k-1) \cup [k+2, l-k-2] \cup (l-k+1, l]$ for all $v \in V(\Gamma_\Delta)$, where $I_2'' = [k+2, l-k-2]$. Then $|I_2''| = l - 2k - 4 < k - 2 < 2$ for $k < 4$.

Since $f(u) = 0$, among the six distinct labels around u (the difference between any pair of them is at least 1), no label is in I_1, at most two labels are in I_2'' (because $|I_2''| < 2$), and at most three labels are in $I_3 = (l-2, l]$ (because $|I_3| < k$ means no two of its labels are for adjacent vertices). So we cannot label all six neighbors of u, a contradiction. ■

We next address $k \geq 4$. The modelling team of Goodwin, Johnston and Marcus [14], obtained the correct values for the integer cases, that is, for integers $k \geq 4$. It is a pity that, due to space limitations, the elegant proof in their contest paper was omitted from the published version! It is the same method we used to prove Proposition 6.8 above.

Moreover, Goodwin *et al.* gave what is equivalent to the correct formula, $\lambda(\Gamma_\Delta; k_1, k_2) = 2k_1 + 6k_2$, for arbitrary integers k_1, k_2 with $k_1 > 6k_2 + 1$. By scaling and continuity, this implies $\lambda(\Gamma_\Delta; k, 1) = 2k + 6$, for all real $k \geq 6$. There appear to be some technical errors in their lower bound proof (quite understandable, since they had just four days to produce their entire paper from scratch!). However, we discovered that if one uses the D-Set Theorem, some small changes will fix their proof. We present below our own verification of the lower bound, which we need more generally for all real $k \geq 4$. We follow this with the much shorter proof for $k \geq 6$ based on the method of Goodwin *et al.* that does not depend on the structure of the triangular lattice, so that it can be used on other graphs, for sufficiently large real k, provided that there is a linear bound for all large integers k.

Proposition 6.11. *For $k \geq 4$ we have $\lambda(\Gamma_\Delta; k, 1) \geq 2k + 6$.*

Proof: Assume for contradiction that $\lambda(\Gamma_\Delta; k, 1) = l < 2k + 6$ for some $k \geq 4$. By the D-Set Theorem, there is an optimal labelling $f \in L(k, 1)(\Gamma_\Delta)$ with span and largest label l and smallest label 0.
Claim 1. *The labelling f cannot use labels in* $[3, k)$.
Proof: If some $f(v) \in [3, k)$, then the labels on the vertices of the C_6 neighboring v are all at least $f(v) + k$. The largest of these labels is then at least $f(v) + k + \lambda(C_6; , k, 1) \geq 3 + k + (k + 3) = 2k + 6 > l$, a contradiction since $\lambda(C_6; , k, 1) = k + 3$ [24, 19]), proving the Claim.

By symmetry, none of the labels in f belongs to $(l - k, l - 3]$. So all labels belong to the union $I_1 \cup I_2 \cup I_3$, where $I_1 = [0, 3)$, $I_2 = [k, l - k]$, and $I_3 = (l - 3, l]$.
Claim 2. *The labelling f cannot use labels in* $[k, k + 1)$.
Proof: Assume some label $f(v) \in [k, k+1)$. At most one of the six vertices next to v has a label in I_1 because any such label is $\leq f(v) - k < 1$. At most three of the six vertices have labels in I_3 as any two must be at least one apart.

First suppose three of these labels are in I_3. They cannot be at adjacent vertices, so suppose they are at vertices v_1, v_3, and v_5, with reference to the graph B_7 in Figure 10. Two of the other labels next to v must belong to $[f(v)+k, l-k]$, so the larger of the two, say it is at v_2, must be at least $f(v) + k + 1 \geq 2k + 1$. Then both $f(v_1)$ and $f(v_3)$ are at least $f(v_2) + k$, and the larger of the two is at least $f(v_2) + k + 1 \geq 3k + 2 \geq 2k + 6 > l$, a contradiction with $k \geq 4$.

Next suppose just two of these labels next to v lie in I_3. The two vertices are not adjacent. There must be at least three labels next to v in $[f(v) + k, l - k]$, and, because this interval has length $< k$, no two of the three are adjacent–say they are at v_1, v_3, v_5. The largest of the three labels is at least $f(v) + k + 2$, and its neighbor with label in I_3 has label at least $f(v) + k + 2 + k \geq 3k + 2$, which is again a contradiction.

Finally, suppose at most one label next to v lies in I_3. Then at least four labels next to v are in $[f(v) + k, l - k]$, so some two are adjacent–but this is impossible since they must differ by at least k (as $(l-k)-(f(v)+k) \leq l-3k < 2 < k$). This proves the Claim.

Hence, f has no labels in $[k, k+1)$ nor, by symmetry, in $(l - k - 1, l - k]$. So all of its labels belong to $I_1 \cup I_2' \cup I_3$, where here $I_2' = [k + 1, l - k - 1]$.
Claim 3. *The labelling f cannot use labels in* $[k + 1, k + 2)$.
Proof: Suppose some $f(v) \in [k + 1, k + 2)$. Then labels used next to v in I_1 are at most $f(v) - k < 2$, so there can be at most two such labels. On the other hand, at most three labels next to v can come from I_3. Then some label used next to v lies in I_2'. But such a label must be at most $l - k - 1$ and at least $f(v) + k \geq 2k + 1 \geq k + 5 > l - k - 1$, a contradiction proving the Claim.

By symmetry, no label of f belongs to $(l - k - 2, l - k - 1]$. Then all of its labels belong to $I_1 \cup I_2'' \cup I_3$, where $I_2'' = [k + 2, l - k - 2]$. Let u be a vertex with $f(u) = 0$.

Then its six neighbors all have labels in $I_2'' \cup I_3$. But I_3 can contain at most three of the labels, as they must be at least one apart from each other. So some three of the labels are in I_2''. However, $(l-k-2)-(k+2) = l-2k-4 < 2$, so I_2'' can contain at most two of the labels, a contradiction, so no such f exists. ■

Here is the shorter proof of the restriction of the Proposition above to $k \geq 6$.

Proposition 6.12. *For $k \geq 6$ we have $\lambda(\Gamma_\Delta; k, 1) \geq 2k+6$.*

Proof: Let us assume the result of Goodwin *et al.* that $\lambda(\Gamma_\Delta; k, 1) \geq 2k+6$ for integers $k \geq 4$. Now consider any non-integer $k > 6$. Let $m = \lceil k \rceil - k$, so that $m \in (0,1)$ and $k+m = \lceil k \rceil \geq 7$. Hence, $\lambda(\Gamma_\Delta; k+m, 1) \geq 2k+2m+6$.

Assume for contradiction that $\lambda(\Gamma_\Delta; k, 1) < 2k+6$. Let f be an optimal labelling in $L(k,1)(\Gamma_\Delta)$ as in the D-Set Theorem, with minimum value 0 at some vertex u and maximum value span(f) at some vertex w. Define a labelling f_1 by $f_1(v) = f(v) + m\lfloor f(v)/k \rfloor$. We can check that $f_1 \in L(k+m,1)(\Gamma_\Delta)$. Further, the minimum value of f_1 is 0, which occurs at u, and its maximum occurs at v, which thus has value $f_1(v) =$ span$(f_1) < (2k+6) + m\lfloor (2k+6)/k \rfloor = 2k+6+2m$ (since $k > 6$ by assumption). This contradicts the lower bound in the previous paragraph. ■

We cannot see how to extend the argument in the last proof to work for k between 4 and 6. It remains to do the lower bound for small k. Similar to the proof of Proposition 6.10, we can show (see [24]):

Proposition 6.13. *For $0 < k \leq \frac{1}{2}$, we have $\lambda(\Gamma_\Delta; k, 1) \geq 2k+3$. Hence $\lambda(\Gamma_\Delta; k, 1) = 2k+3$ for $0 \leq k \leq \frac{3}{7}$.* ■

This completes the proof of Theorem 3.1.

7 The Proof of the Square Lattice Theorem 4.1

We begin by establishing the claimed upper bounds on $\lambda(\Gamma_\Box; k, 1)$ for reals $k \geq 0$. In many cases, there is an explicit construction based on a modular construction, in which a particular matrix of labels is used for a rectangle of lattice points and then repeated over and over. We define a doubly periodic labelling of the square lattice by an $m \times n$ labelling matrix $A := [a_{i,j}]$, such that we label point (i,j) by $a_{m-(j \bmod m),\,(i \bmod n)+1}$, where i, j are integers.

Proposition 7.1. *For $0 \leq k \leq \frac{1}{2}$, we have $\lambda(\Gamma_\Box; k, 1) \leq k+3$.*

Proof: Starting from an optimal $L(0,1)$-labelling and shifting up some labels by k, in order to satisfy the $L(k,1)$ conditions, we constructed with the following labelling matrix that attains the upper bound:

$$A = \begin{bmatrix} 0 & k & 1 & k+1 \\ k+3 & 2 & k+2 & 3 \\ 1 & k+1 & 0 & k \\ k+2 & 3 & k+3 & 2 \end{bmatrix}. \quad ■$$

Next, applying the result above at $k = 1/2$, Lemma 2.2 yields this upper bound for larger k:

Proposition 7.2. *For $\frac{1}{2} \leq k \leq \frac{4}{7}$, we have $\lambda(\Gamma_\Box; k, 1) \leq 7k$.* ■

Van den Heuvel, Leese and Shepherd [23] gave a circular integer labelling result which is helpful for our real number labellings, as it suggests some arithmetic progression labellings that turn out to be optimal for our problem:

Proposition 7.3. *We have*
$\lambda(\Gamma_\Box; 1, 1) = 4$,
$\lambda(\Gamma_\Box; 2, 1) \leq 6$,
$\lambda(\Gamma_\Box; 3, 1) \leq 8$, *and*
$\lambda(\Gamma_\Box; 3, 2) \leq 11$.

Proof: From [23], we have these labellings:
$\lambda(\Gamma_\Box; 1, 1) \leq 4$ by labelling f with $f(i, j) = (i + 2j) \bmod 5$
$\lambda(\Gamma_\Box; 2, 1) \leq 6$ by labelling f with $f(i, j) = (2i + 3j) \bmod 7$
$\lambda(\Gamma_\Box; 3, 1) \leq 8$ by labelling f with $f(i, j) = (3i + 4j) \bmod 9$
$\lambda(\Gamma_\Box; 3, 2) \leq 11$ by labelling f with $f(i, j) = (3i + 5j) \bmod 12$.
It is easy to show $\lambda(\Gamma_\Box; 1, 1) \geq 4$. ■

Applying the preceding two propositions and Lemma 2.2, we have the following upper bounds.

Proposition 7.4. *We have* $\lambda(\Gamma_\Box; k, 1) \leq \begin{cases} 4 & \text{if } \frac{4}{7} \leq k \leq 1 \\ 4k & \text{if } 1 \leq k \leq \frac{5}{3} \\ 6 & \text{if } \frac{5}{3} \leq k \leq 2 \\ 3k & \text{if } 2 \leq k \leq \frac{8}{3} \\ 8 & \text{if } \frac{8}{3} \leq k \leq 3 \end{cases}$. ■

The upper bounds in the proposition above are weaker than the known lower bounds for $\frac{4}{3} < k < \frac{5}{3}$. Let us consider one value in this gap, $k = 11/8$. By Proposition 7.4, we get the upper bound $\lambda(\Gamma_\Box; \frac{11}{8}, 1) \leq \frac{11}{2}$, so by the Scaling Lemma, $\lambda(\Gamma_\Box; 11, 8) \leq 44$. To determine whether this is best-possible, we searched for a better labelling: We managed to construct a $L(11, 8)$-labelling based on a matrix A in which the entries are elements of the D-set in $[0, 43]$. Since 43 can be expressed in terms of 11 and 8 in just one way, $43 = 11 + 4 \times 8$, we were able to extend this matrix labelling to cases in the range $\frac{4}{3} \leq k \leq \frac{3}{2}$, as given in the following proposition. We next took the resulting labelling at $k = \frac{3}{2}$, and found a way to extend it to the range $\frac{3}{2} \leq k \leq \frac{5}{3}$ in a way that maintains the order of the labels, while expanding their pairwise differences, to maintain feasibility as k grows, giving Proposition 7.6. Notice that these formulas for $\lambda(\Gamma_\Box; k, 1)$ around $k = 11/8$ are not of the simple form ck for some c, so we could not simply apply Lemma 2.2.

Proposition 7.5. *For $\frac{4}{3} \leq k \leq \frac{3}{2}$, we have $\lambda(\Gamma_\Box; k, 1) \leq k + 4$.*

Proof: The upper bound is attained by the following labelling matrix:

$$\begin{bmatrix} 5 & k & 4 & 0 & 3 & k{+}4 & 2 & k{+}3 & 1 & k{+}2 & 0 & k{+}1 \\ 0 & 3 & k{+}4 & 2 & k{+}3 & 1 & k{+}2 & 0 & k{+}1 & 5 & k & 4 \\ 2 & k{+}3 & 1 & k{+}2 & 0 & k{+}1 & 5 & k & 4 & 0 & 3 & k{+}4 \\ k{+}2 & 0 & k{+}1 & 5 & k & 4 & 0 & 3 & k{+}4 & 2 & k{+}3 & 1 \end{bmatrix}. \quad \blacksquare$$

Proposition 7.6. *For $\frac{3}{2} \le k \le \frac{5}{3}$, we have $\lambda(\Gamma_\Box; k, 1) \le 3k + 1$.*

Proof: The upper bound is attained by the following labelling matrix:

$$\begin{bmatrix} 2k+2 & k & 2k+1 & 0 & 2k & 3k+1 & 2 & 3k & 1 & k+2 & 0 & k+1 \\ 0 & 2k & 3k+1 & 2 & 3k & 1 & k+2 & 0 & k+1 & 2k+2 & k & 2k+1 \\ 2 & 3k & 1 & k+2 & 0 & k+1 & 2k+2 & k & 2k+1 & 0 & 2k & 3k+1 \\ k+2 & 0 & k+1 & 2k+2 & k & 2k+1 & 0 & 2k & 3k+1 & 2 & 3k & 1 \end{bmatrix}. \quad \blacksquare$$

For larger k, we first adapt the construction given by Calamoneri for integers k_1, k_2 with $3k_2 \le k_1 \le 4k_2$. We then present a simple matrix $L(k,1)$-labelling that is optimal for all $k \ge 4$.

Proposition 7.7. *For $3 \le k \le 4$ we have $\lambda(\Gamma_\Box; k, 1) \le 2k + 2$.*

Proof: Adapting the construction in [6], the upper bound is attained by the $L(k,1)$-labelling matrix:

$$A = \begin{bmatrix} 2k+2 & k & 2k+1 & 2 & 2k & 1 & k+2 & 0 & k+1 \\ k+2 & 0 & k+1 & 2k+2 & k & 2k+1 & 2 & 2k & 1 \\ 2 & 2k & 1 & k+2 & 0 & k+1 & 2k+2 & k & 2k+1 \end{bmatrix}. \quad \blacksquare$$

Proposition 7.8. *For $k \ge 0$ we have $\lambda(\Gamma_\Box; k, 1) \le k + 6$.*

Proof: The upper bound is attained by the following labelling matrix:

$$A = \begin{bmatrix} 0 & k+3 & 1 & k+4 \\ k+6 & 2 & k+5 & 3 \\ 1 & k+4 & 0 & k+3 \\ k+5 & 3 & k+6 & 2 \end{bmatrix}. \quad \blacksquare$$

We now consider the lower bounds to complete the proof of the formulas. It is helpful to compare our graph to T_4, the regular infinite tree of degree 4 discussed in Section 2 By Theorem 2.7 we get that for all $k \ge 0$, $\lambda(\Gamma_\Box; k, 1) \ge \lambda(T_4; k, 1)$. From the values of $\lambda(T_4; k, 1)$ presented in Theorems 2.4 and 2.5, we obtain the claimed values of $\lambda(\Gamma_\Box; k, 1)$ for all k outside the interval $[\frac{5}{2}, 3]$. In this remaining interval, we must improve the lower bound on $\lambda(\Gamma_\Box; k, 1)$. In view of Lemma 2.2, all that remains to prove the theorem is to establish the lower bound at $k = 8/3$:

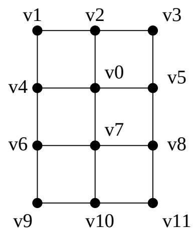


Figure 11: The Subgraph B_{12} of the Square Lattice

Proposition 7.9. *We have* $\lambda(\Gamma_\square; 8, 3) \geq 24$. *Consequently, for* $2 \leq k \leq 3$, *we have* $\lambda(\Gamma_\square; k, 1) \geq \begin{cases} 3k & \text{if } 2 \leq k \leq \frac{8}{3} \\ 8 & \text{if } \frac{8}{3} \leq k \leq 3 \end{cases}$.

Proof: The second statement follows from the first by Lemma 2.2.

Assume for contradiction that the first statement fails. Then there exists a labelling $f \in L(8,3)(\Gamma_\square)$ with all labels in $\{0, \ldots, 23\}$. The series of claims that follows restricts the labels f one can use until we find that no such f can exist at all, proving the proposition.

Let $v_0 = (i_0, j_0) \in V(\Gamma_\square)$. Let B_{12} be the induced subgraph as in Figure 11.

Claim 1. *The labelling f cannot use label 7 or 16.*

Proof: Assume $f(v_0) = 16$. Since no label can exceed 23, the four distinct labels around v_0 are each $\leq f(v_0) - 8 = 8$, which is impossible since any two must be at least 3 apart. By the Symmetry Lemma, labelling f does not use the complementary label $23 - 16 = 7$, and the Claim follows.

Claim 2. *The labelling f cannot use label 8 or 15.*

Proof: Assume some $f(v_0) = 8$. The four labels around v_0 are each $\geq f(v_0) + 8 = 16$ or $\leq f(v_0) - 8 = 0$, hence are 0 or ≥ 17 (because by Claim 1, f cannot use 16). Suppose they are labels $x < y < z < w$. Since the difference between any pair of the four labels is ≥ 3, it must be that $x = 0, y = 17, z = 20, w = 23$. Suppose without loss of generality that $f(v_7) = y = 17$. Since $f(v_7) + 8 = 25$ is too large, it must be that the neighboring labels $f(v_6), f(v_8), f(v_{10})$ are all $\leq f(v_7) - 8 = 9$, and hence, all $\leq f(v_0) - 3 = 8 - 3 = 5$. But this is impossible since the difference between any pair of the three labels must be at least 3. By symmetry, we can also exclude 15. This proves the Claim.

Now f has no label $7, 8, 15, 16$. The proofs of Claims 3 and 4 are similar to the proof of Claim 2, so we omit the details.

Claim 3. *The labelling f cannot use label 9 or 14.*

Claim 4. *The labelling f cannot use label 11 or 12.*

By the D-Set Theorem, there exists optimal labelling $f^* \in L(8,3)(\Gamma_\square)$ with smallest label 0 and all labels in $D_{8,3} \cap [0, 23] = \{0, 3, 6, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. Applying the Claims above to f^*, we find that $f^*(v) \in \{0, 3, 6, 17, 18, 19, 20, 21, 22, 23\}$ for all $v \in V(\Gamma_\square)$.

Let $f(v_0) = 0$. The four labels around v_0 are each $\geq f(v_0) + 8 = 8$. Their labels belong to $\{17, 18, 19, 20, 21, 22, 23\}$, a contradiction since the difference between any pair of them is ≥ 3. Thus, it must be that $\lambda(\Gamma_\square; 8, 3) \geq 24$. ■

This completes the proof of the formulas for the square lattice, Theorem 4.1.

8 The Proof of the Hexagonal Lattice Theorem 5.1

We will find the upper bound on $\lambda(\Gamma_H; k, 1)$, $k \geq 0$, by constructions and Lemma 2.2. One construction method is to tile the whole lattice by a labelled parallelogram described by a matrix of labels. We define a doubly periodic labelling of the Hexagonal Lattice by an $m \times n$ labelling matrix $A := [a_{i,j}]$, for m, n even, such that we label point (i, j) by We define a doubly periodic labelling of the square lattice by an $m \times n$ labelling matrix $A := [a_{i,j}]$, such that we label point (i, j) by $a_{m-(j \bmod m),\ (i \bmod n)+1}$, where i, j are integers. For example, the following labelling (see Figure 12) is defined by the labelling matrix A, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{bmatrix}$$

Then Figure 12 shows how the labels are assigned, where $a_{4,1}$ is at the vertex with coordinates $(0, 0)$ in the hexagonal lattice. The whole lattice is tiled with copies of the 4×6 tile as shown:

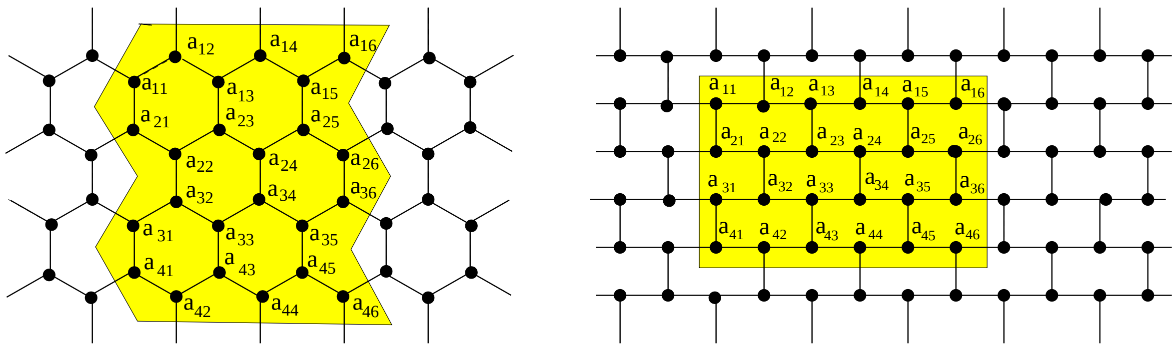


Figure 12: The Doubly Periodic Labelling by Matrix A

Proposition 8.1. *For $0 \leq k \leq \frac{1}{2}$, we have $\lambda(\Gamma_H; k, 1) \leq k + 2$.*

Proof: We use the labelling matrix below, also shown in Figure 13, with the values a, b, c taken to be $k, k+1, k+2$, respectively:

$$A = \begin{bmatrix} 0 & a & 1 & b & 2 & c \\ b & 2 & c & 0 & a & 1 \end{bmatrix}$$

Incidentally, this labelling was obtained by doing a first-fit labelling on one row, then on the next row, and so on. ■

We have $\lambda(\Gamma_H; \frac{1}{2}, 1) \leq \frac{5}{2}$. By Lemma 2.2, it follows that:

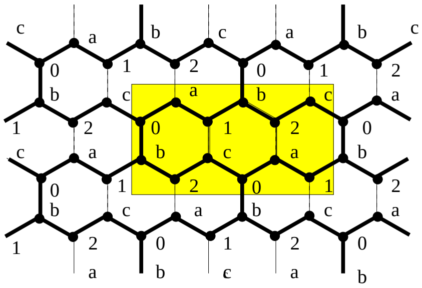


Figure 13: Optimal $L(k,1)$-labelling of Γ_H for $0 \le k \le \frac{1}{2}$ or $k \ge 3$.

Proposition 8.2. *For $\frac{1}{2} \le k \le \frac{3}{5}$, we have $\lambda(\Gamma_H; k, 1) \le 5k$.* ■

Next we consider $k = 1$:

Proposition 8.3. *We have $\lambda(\Gamma_H; 1, 1) \le 3$. Hence, $\lambda(\Gamma_H; k, 1) \le \begin{cases} 3 & \text{if } \frac{3}{5} \le k \le 1 \\ 3k & \text{if } 1 \le k \le \frac{5}{3} \end{cases}$*

Proof: Because of Lemma 2.2, it is enough to prove the upper bound at $k = 1$.

We will prove $\lambda(\Gamma_H; 1, 1) \le 3$ by using either of the following labelling matrices. Each was obtained by a first-fit labelling process, doing one row at a time. (See Figure 14.)

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 3 \\ 1 & 3 & 0 & 2 \\ 0 & 3 & 1 & 2 \end{bmatrix} \qquad \text{or} \qquad A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 3 & 0 & 2 \end{bmatrix} \quad ■$$

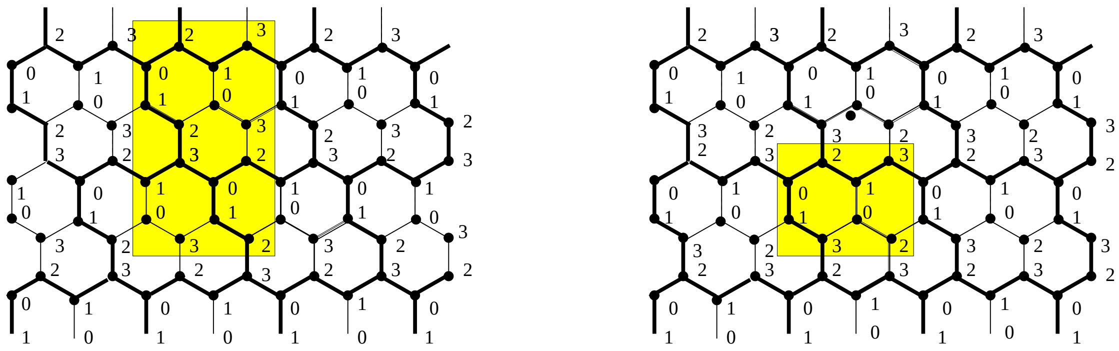


Figure 14: Optimal $L(1,1)$-labelling of Γ_H

Proposition 8.4. *For $2 \le k \le 3$, we have $\lambda(\Gamma_H; k, 1) \le 2k + 1$. For $\frac{5}{3} \le k \le 2$, we have $\lambda(\Gamma_H; k, 1) \le 5$.*

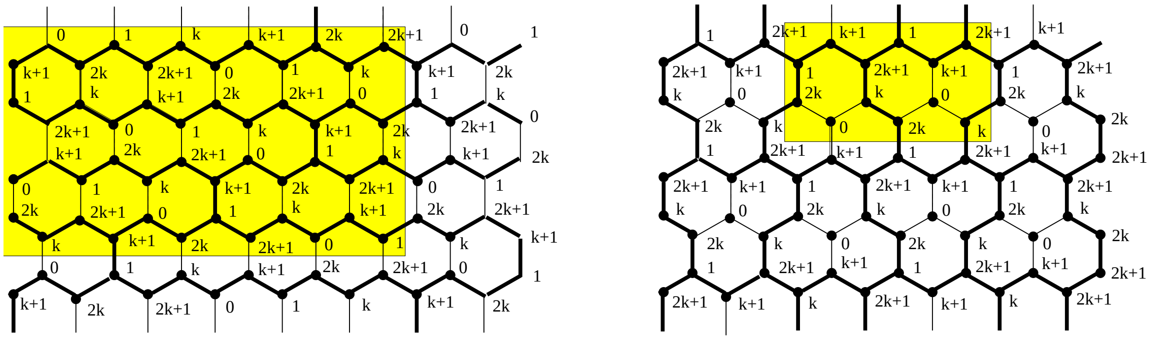


Figure 15: Optimal $L(k,1)$-labellings of Γ_H for $2 \le k \le 3$.

Proof: The second statement follows immediately from the first at $k = 2$. For $2 \le k \le 3$, one can prove $\lambda(\Gamma_H; k, 1) \le 2k+1$ by the matrix labelling with entries shown in Figure 15 (left). A simpler construction can be obtained by adapting a construction of Calamoneri [6], originally given for the corresponding integer labelling. This gives the following matrix labelling, shown in Figure 15 (right):

$$A = \begin{bmatrix} 1 & k+1 & 2k+1 & 1 & k+1 & 2k+1 \\ 2k & 0 & k & 2k & 0 & k \end{bmatrix}. \quad \blacksquare$$

Next we treat large k:

Proposition 8.5. *For $k \ge 3$, we have $\lambda(\Gamma_H; k, 1) \le k+4$.*

Proof: Following the construction in [6] for the corresponding integer labelling, we again have the matrix labelling as in Figure 13, where this time $a = k+4, b = k+3, c = k+2$:

$$A = \begin{bmatrix} 0 & a & 1 & b & 2 & c \\ b & 2 & c & 0 & a & 1 \end{bmatrix}. \quad \blacksquare$$

We next verify the lower bounds. By Theorem 2.7 we get that for all $k \ge 0$, $\lambda(\Gamma_\square; k, 1) \ge \lambda(T_3; k, 1)$. From the values of $\lambda(T_3; k, 1)$ presented in Theorems 2.3 and 2.5, we obtain the claimed values of $\lambda(\Gamma_\square; k, 1)$ for all k outside the interval $(\frac{3}{2}, 2)$. In this remaining interval, we must improve the lower bound on $\lambda(\Gamma_\square; k, 1)$. In view of Lemma 2.2, all that is needed to complete the proof is to establish the lower bound at $k = 5/3$:

Proposition 8.6. *We have $\lambda(\Gamma_H; 5, 3) \ge 15$. Hence,* $\lambda(\Gamma_H; k, 1) \le \begin{cases} 3k & \text{if } 1 \le k \le \frac{5}{3} \\ 5 & \text{if } \frac{5}{3} \le k \le 2 \end{cases}$

Proof: It suffices to prove the first statement, due to Lemma 2.2. We will show $\lambda(\Gamma_H; 5, 3) \ge 15$.

Assume otherwise, $\lambda(\Gamma_H; 5, 3) < 15$. Then there exists a $L(5,3)$-labelling f with all labels in the set $\{0, \ldots, 14\}$.

Claim 1. *The labelling f cannot use labels 4 or 10.*

Proof: Assume $f(v) = 4$ for some $v \in V(\Gamma_H)$. The three distinct labels around v are $\geq f(v) + 5 = 9$. Suppose they are labels $x_1 < x_2 < x_3$. Since any pair of the three labels differ by at least 3 (because they are at distance two each other), one of them is ≥ 15, a contradiction. By the Symmetry Lemma, f cannot use label $14 - 4 = 10$. This proves the Claim.

Claim 2. *The labelling f cannot use labels 5 or 9.*

Proof: Assume $f(v) = 5$ for some $v \in V(\Gamma_H)$. The three labels around v are $\leq f(v) - 5 = 0$ or $\geq f(v) + 5 = 10$. But 10 is excluded by the previous Claim. Since any pair of the three labels differ by at least 3 it must be that the three labels used are 0, 11, 14. Then the three neighbors of the label 11 are each $\leq 11 - 5 = 6$ and any two are at least 3 apart, so they need to be 0, 3, and 6. But this is a contradiction since one of them is $f(v) = 5$. By symmetry, we must also exclude label 9, and the Claim follows.

Now f has no label $4, 5, 9, 10$. The proofs of Claims 3 and 4 are similar to the proof of Claim 2, so we omit the details.

Claim 3. *The labelling f cannot use label* 7*.*

Claim 4. *The labelling f cannot use labels* $1, 2, 12$*, or 13.*

Now all labels of f belong to $\{0, 3, 6, 8, 11, 14\}$, call this set L.

Claim 5. *The labelling f cannot use labels* 3 *or 11.*

Proof: Assume $f(v) = 3$ for some $v \in V(\Gamma_H)$. The three labels around v are $\geq f(v) + 5 = 3 + 5 = 8$. They are $8, 11, 14$ as in Figure 16. The three neighbors of the label 11 are ≤ 6, with one of them $f(v) = 3$ and the others are $0, 6$. By the separation conditions and set L, we have $c, d \in \{0, 14\}$. We have two cases.

Case 1. $a = 0, b = 6$. Since $a = 0$, then $c = 14$. We cannot find a feasible label g in L, a contradiction.

Case 2. $a = 6, b = 0$. Since $b = 0$, then $e \in \{6, 8\}, f = 0$, so that $d = 14$. We cannot find a feasible label h in L, a contradiction proving the Claim.

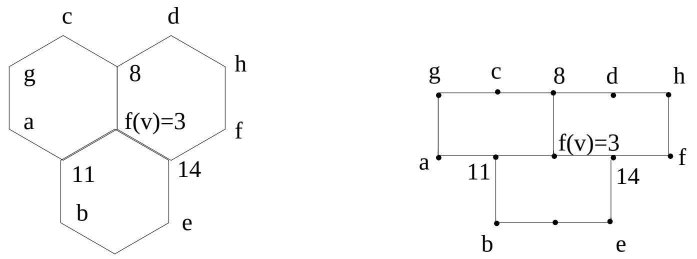


Figure 16: The $L(5, 3)$-labelling of a Subgraph of Γ_H

Now all labels of f belong to $\{0, 6, 8, 14\}$. We cannot label the induced subgraph $K_{1,3}$, a contradiction. ■

This completes the proof of the span formulas for Γ_H.

9 Concluding Remarks

Despite considerable effort, we failed to complete the determination of all values $\lambda(\Gamma_\Delta; k, 1)$ for $\frac{1}{3} < k < \frac{4}{5}$. We believed that our upper bounds would be the correct values for $k \in [1/3, 1/2]$, while the line $5k + 2$ would be correct for $k \in [1/2, 4/5]$. However, while preparing our revision of this manuscript, Král' and Škoda (building on our earlier version in circulation) found this is not entirely the case [27]. They completed the determination of $\lambda(\Gamma_\Delta; k, 1)$. Altogether, the graph has eleven linear pieces in the interval $[0, 1]$.

As with the graphs of $\lambda(\Gamma_\Box; k, 1)$ and $\lambda(\Gamma_H; k, 1)$, the graph of $\lambda(\Gamma_\Delta; k, 1)$ has the property that successive linear pieces alternately turn up and down, that is, though continuous and nondecreasing in k, the concavity keeps alternating between up and down. We continue to be puzzled why this is so.

It is natural now to extend this investigation to conditions at distance three, where very little is known. One definite result in this direction is given by Bertossi, Pinotti, and Tan [3], who determined that $\lambda(\Gamma_\Delta; 2, 1, 1) = 11$. Their construction that achieves the optimal value, 11, can be described by a labelling matrix:

$$A = \begin{bmatrix} 0 & 10 & 2 & 6 & 1 & 9 \\ 4 & 8 & 5 & 11 & 3 & 7 \\ 2 & 6 & 1 & 9 & 0 & 10 \\ 5 & 11 & 3 & 7 & 4 & 8 \\ 1 & 9 & 0 & 10 & 2 & 6 \\ 3 & 7 & 4 & 8 & 5 & 11 \end{bmatrix}.$$

On the other hand, it is easy to find 12 vertices in Γ_Δ such that the maximum distance is three, and so $\lambda(\Gamma_\Delta; 1, 1, 1) \geq 11$. It follows that $\lambda(\Gamma_\Delta; k, 1, 1) = 11$ for $1 \leq k \leq 2$.

Several papers in engineering consider labellings with conditions at distance at most p with $k_1 = k \geq k_2 = k_3 = \cdots = k_p = 1$. Bertossi *et al.* [3] and then Panda *et al.* [31] independently obtained a lower bound for the square lattice, $\lambda(\Gamma_\Box; 1, 1, \ldots, 1) \geq \lfloor \frac{p^2+2p}{2} \rfloor$. This problem corresponds to integer channel assignments such that identical channels must be at least distance $p+1$ apart in the lattice. Investigating graph models of wireless networks, Dubhashi *et al.* [9] present bounds on the minimum span for $L(2, 1, 1, \cdots, 1)$-labelling of the p-dimensional square lattice (grid), in which $V(G) = \mathbb{Z}^p$, and two vertices, say $(x_1, x_2, \ldots, x_p)$ and $(y_1, y_2, \ldots, y_p)$, are joined by an edge whenever $\sum_{i=1}^{p} |x_i - y_i| = 1$.

We are continuing this project by seeking to describe *all* optimal $L(k, 1)$-labellings of the three regular lattices, and by searching for optimal labellings with nice symmetry properties, such as being periodic or doubly periodic.

For further progress on the general theory of real-number graph labellings, the reader is recommended to check the paper of Král' [26]. For an overview of the recent progress and state of the theory, refer to the recent survey [20].

The authors need to acknowledge the encouragement and sharing of ideas by their international research colleagues during this long-running project. Particular thanks go to Tiziana Calamoneri, Gerard Chang, John Georges, Dan Král', Jan Kratochvíl, Daphne Liu, David Mauro, Fred Roberts, and Roger Yeh. We are indebted to the dedicated and diligent referees for their many thoughtful editorial suggestions.

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