Prof. Girardi	Math 555/704i.001	03.28.08	Spring 2008	Exam 2

MARK BOX		
PROBLEM	POINTS	
FB (1–15)	30	
T/F (1–19)	19	
choice 1	17	
choice 2	17	
choice 3	17	
TOTAL	100	

**** Do 3 (and only 3) of problems 1 – 5. ****

I picked the 3 problems: ____________________

Name (printed): ____________________

INSTRUCTIONS:

(1) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.

(2) For the proof problems, write a **neat formal** proof on the lined paper provided. Start each new problem on a new page and do not write on the back of the lined paper. You do NOT have to recopy the statement of the problem and do not have to work the problems in order. Do your scratch work on the scrap paper provided and do not hand it in.

(3) You may use, without proving, a textbook's or class's: Theorem, Corollary, Lemma, Example, or Exercise (unless you are asked to specifically prove that specific item).

(4) This is a closed book/notes exam covering (from *Introduction to Real Analysis*, 2nd ed., by Stoll): Chapters 4 and 5 .

NOTATION for throughout exam:

- I is an **INTERVAL** in $\mathbb{R}$. and E is a **SUBSET** of $\mathbb{R}$.
- $\text{Int}(E)$ denotes the set of interior points of E.
  $\overline{E}$ denotes the closure of E.
  E' denotes the set of limit points of E.
- *iff* is short for *if and only if*

Problem Inspiration:

(1) homework problem Ch 4 Misc 3
(2) homework 4.3.11
(3) Thm 5.2.2 from textbook
(4) homework Ch 5 Misc 6
(5) Thm 5.2.13 from textbook

FB. Fill in the blanks. When it says, *by definition*, be sure to give the definition and not some equivalent formulation.

FB.1 Let $f\colon E \to \mathbb{R}$ and $p \in E'$ and $L \in \mathbb{R}$. By definition, $\lim_{x\to p} f(x) = L$ iff

____

FB.2 Let $f\colon E \to \mathbb{R}$ and $p \in E$. By definition, f is continuous at p iff (should be an ε in there)

____

FB.3 Let $f\colon E \to \mathbb{R}$. By the topological characterization of continuity, f is continuous on E iff $f^{-1}(V)$ is ______________ in E for each ______________ subset V of $\mathbb{R}$.

FB.4 Let K be a **compact** subset of $\mathbb{R}$ and $f\colon K \to \mathbb{R}$ be continuous. Then there exists $p, q \in K$ such that _________ $\le f(x) \le$ _________ for each ____________ .

FB.5 Let K be a **compact** subset of $\mathbb{R}$ and $f\colon K \to \mathbb{R}$ be continuous. Two BIG theorems from class say that the set $f(K)$ is ________________ and f is ________________ .

FB.6 Let $f\colon [a, b] \to \mathbb{R}$ be continuous and $f(a) < \gamma < f(b)$. Then the Intermediate Value Theorem says that ____ .

FB.7 Let $f\colon E \to \mathbb{R}$. By definition, f is uniformly continuous on E iff

____

FB.8 Let $f\colon E \to \mathbb{R}$. By definition, f is Lipschitz on E (with Lipschitz constant at most $M > 0$) iff ____

FB.9 Let $f\colon I \to \mathbb{R}$ and $p \in I$. The derivative of f at p, denoted by $f'(p)$, is defined to be

$f'(p) =$ ________________________________ provided the limit exists.

FB.10 Let $f\colon I \to \mathbb{R}$ and $p \in I$. If $I \cap$ ____________ $\neq \emptyset$, then the right derivative of f at p, denoted by $f'_+(p)$, is defined to be

$f'_+(p) =$ ________________________________ provided the limit exists.

FB.11 Let $f\colon [a, b] \to \mathbb{R}$ be continuous. If f has a local extremum at $p \in (a, b)$, then either _______________________ or _______________________.

FB.12 Finish the statement of **Rolle's Theorem**.
Let $f\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

____.

FB.13 Finish the statement of the **Mean Value Theorem (MVT)**.
Let $f\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

____.

FB.14 Finish the statement of the **Cauchy's Mean Value Theorem**.
Let $f, g\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

____.

FB.15 Let $f\colon (a, b) \to \mathbb{R}$. By definition, f is convex on (a, b) iff

____.

T/F. If the statement is true, then circle T. If the statement is false, then circle F.

TF.1 **T F** Let $f\colon E \to \mathbb{R}$ and $p \in E \cap E'$. Then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

TF.2 **T F** Let $f\colon E \to \mathbb{R}$ and $p \in E \cap E'$. Then f is continuous at p iff $\lim_{n \to \infty} f(p_n) = f(p)$ for each sequence $\{p_n\}$ in E with $p_n \to p$.

TF.3 **T F** Let $f\colon E \to \mathbb{R}$ and $p \in E$ but $p \notin E'$. Then f is continuous at p.

TF.4 **T F** Define $f\colon (0,1) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x \text{ is rational with } x = \frac{m}{n} \text{ in lowest terms.} \end{cases}$$

Then f is discontinuous at each rational in $(0,1)$ and continuous at each irrational in $(0,1)$.

TF.5 **T F** If $f\colon [0,1] \to [0,1]$ is continuous, then there exists $c \in [0,1]$ such that $f(c) = c$.

TF.6 **T F** Let $f\colon E \to \mathbb{R}$. If f is Lipschitz, then f is uniformly continuous.

TF.7 **T F** Let $f\colon E \to \mathbb{R}$. If f is uniformly continuous, then f is Lipschitz.

TF.8 **T F** The set of discontinuities of a monotone function on an open interval is finite.

TF.9 **T F** The set of discontinuities of a monotone function on an open interval is at most countable.

TF.10 **T F** Let $f\colon I \to \mathbb{R}$ and $p \in I$. If f is differentiable at p, then f is continuous at p.

TF.11 **T F** Let $f\colon I \to \mathbb{R}$ and $p \in I$. If f is continuous at p, then f is differentiable at p.

TF.12 **T F** Let $f\colon I \to \mathbb{R}$ be differentiable on I. If $f'(x) \geq 0$ for each $x \in I$, then f is monotone increasing on I.

TF.13 **T F** Let $f\colon I \to \mathbb{R}$ be differentiable on I and $c \in \text{Int}(I)$. If $f'(c) > 0$, then there is a $\delta > 0$ so that f is monotone increasing on $(c - \delta, c + \delta)$.

TF.14 **T F** Let $f\colon I \to \mathbb{R}$ be differentiable on I and $c \in \text{Int}(I)$. If $f'(c) > 0$ and f' is continuous at c, then there is a $\delta > 0$ so that f is monotone increasing on $(c - \delta, c + \delta)$.

TF.15 **T F** Let $f\colon (a,b) \to \mathbb{R}$ be convex. Let $a < x_1 < x_2 < x_3 < b$. Then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

(roughly speaking, the slope of the secant lines are increasing).

TF.16 **T F** Let $f\colon (a,b) \to \mathbb{R}$ be convex. Then f is continuous on (a,b).

TF.17 **T F** Let $f\colon (a,b) \to \mathbb{R}$ be convex and $x \in (a,b)$. Then the right derivative and the left derivative of f at x exists.

TF.18 **T F** Let $f\colon (a,b) \to \mathbb{R}$ be convex. Then f is differentiable on (a,b).

TF.19 **T F** Let $f\colon (a,b) \to \mathbb{R}$ be twice differentiable on (a,b). Then f is convex on (a,b) if and only if $f''(x) \geq 0$ for each $x \in (a,b)$.

Do 3 (and only 3) of problems 1 – 5.

1. § 4.2: Continuous Functions

Let $I = [a, b]$ and $f \colon I \to \mathbb{R}$ be bounded. For $A \subset I$ and $p \in I$, define

$$\begin{aligned} \operatorname{osc}(f, A) &:= \sup\{|f(x) - f(y)| : x, y \in A\} \\ \omega(f, p) &:= \lim_{\delta \to 0^+} \operatorname{osc}(f, N_\delta(x) \cap I) \ . \end{aligned}$$

Clearly, $\operatorname{osc}(f, A)$ and $\omega(f, p)$ are in $[0, \infty)$.

1a. Show that f is continuous at $p \in I$ if and only if $\omega(f, p) = 0$.

1b. Fix $r > 0$. Show that the set $\{x \in I \colon \omega(f, x) < r\}$ is open relative to I.

ps: Use the definition of osc (f, A) as given above (which is from the textbook) and not the (equivalent) definition from class.

2. § 4.3: Uniform Continuity

Let $-\infty \leq a < c < b \leq \infty$ and $f \colon (a, b) \to \mathbb{R}$ be continuous on (a, b). Show that if f is uniformly continuous on (a, c) and also on (c, b), then f is uniformly continuous on (a, b).

3. § 5.1: Derivative

Let $f \colon I \to \mathbb{R}$ and $p \in \operatorname{Int}(I)$. Show that if f has a local extremum at p and f is differentiable at p, then $f'(p) = 0$.

REMARK. You may use just the definitions of local extremum and of the right-&-left derivatives.

4. § 5.2: MVT

Let $f \colon (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Let $\{s_n\}$ and $\{t_n\}$ be sequences in (a, b) with $s_n < c < t_n$ and $\lim_{n\to\infty}(t_n - s_n) = 0$. Show that

$$\lim_{n \to \infty} \frac{f(t_n) - f(s_n)}{t_n - s_n} = f'(c) \ .$$

Hint:

$$\tfrac{f(t_n) - f(s_n)}{t_n - s_n} - f'(c) = \tfrac{t_n - c}{t_n - s_n} \tfrac{f(t_n) - f(c)}{t_n - c} + \tfrac{c - s_n}{t_n - s_n} \tfrac{f(c) - f(s_n)}{c - s_n} - \tfrac{t_n - c}{t_n - s_n} f'(c) - \tfrac{c - s_n}{t_n - s_n} f'(c) \ .$$

5. § 5.2: MVT

Let $f \colon [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and $f'(a) < \lambda < f'(b)$. Show that there exists a $c \in (a, b)$ so that $f'(c) = \lambda$.

HINT: We are not assuming that f' is continuous so we cannot apply the usual IVT to f'. However, you can use, without proving, problem 3 from this exam.