Prof. Girardi	Math 555/704i.001	02.20.08	Spring 2008	Exam 1 - Part 2

MARK BOX		
PROBLEM	POINTS	
Part 1	50	
choice 1/part 2	17	
choice 2/part2	17	
choice 3/part2	17	
TOTAL	100	

**** Do 3 (and only 3) of the 6 problems. ****

I picked the 3 problems: ____________________

Name (printed): ____________________________

INSTRUCTIONS:

(1) Write a **neat formal** proof on the lined paper provided. Start each new problem on a new page and do not write on the back of the lined paper. You do NOT have to recopy the statement of the problem and do not have to work the problems in order. But on each page, in the upper right hand cornor, put the problem number. Do your scratch work on the scrap paper provided and do not hand it in.

(2) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.
Hand in this sheet of paper along with your proofs on the lined paper.

(3) You may use, without proving, a textbook's or class's: Theorem, Corollary, Lemma, Example, or Exercise.

(4) This is a closed book/notes exam covering (from *Introduction to Real Analysis*, 2nd ed., by Stoll): § 2.5, 2.6, 2.7, 3.1, 3.2, 3.3, 4.1 .

NOTATION for throughout exam:

- $\{x_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ are sequences from $\mathbb{R}$. $\sum_{n=1}^{\infty} a_n$ is an infinite series in $\mathbb{R}$.
- $\overline{E}$ denotes the closure of E. *iff* is short for *if and only if*

1. <u>§ 2.5: $\overline{\lim}$ and $\underline{\lim}$</u> Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence from $\mathbb{R}$. Consider its corresponding sequence $\{a_n\}_{n=1}^{\infty}$ of averages, so for each $n \in \mathbb{N}$

$$a_n := \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i \ . \tag{1.1}$$

1a. Show that $\overline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} x_n$.

Hints: Let $\beta := \overline{\lim}_{n\to\infty} x_n$. Since $\{x_n\}$ is bounded, There exists $B \in \mathbb{R}$ so that $x_n < B$ for each $n \in \mathbb{N}$. If $n > N_1$ then

$$a_n = \frac{1}{n}\sum_{i=1}^{N_1} x_i + \frac{1}{n}\sum_{i=N_1+1}^{n} x_i \le \frac{N_1}{n} B + \frac{n-N_1}{n} \max\{x_{N_1+1}, x_{N_1+2}, \ldots, x_n\} \ .$$

Also useful is (homework) Exercise 2.5.4, which is the following. Let $\beta \in \mathbb{R}$. If for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ so that $a_n \le \beta + 17\varepsilon$ for each $n \ge N_0$, then $\overline{\lim}_{n\to\infty} a_n \le \beta$.

1b. Carefully show that $\underline{\lim}_{n\to\infty} x_n \le \underline{\lim}_{n\to\infty} a_n$ by using part 1a and homework # 2.5.6b, which is the following. If $\{b_n\}$ is bounded, then $\underline{\lim}_{n\to\infty}(-b_n) = -\overline{\lim}_{n\to\infty} b_n$. Do NOT give an ε-N argument.

1c. Show that if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} a_n = x$.

2. § 2.6: Cauchy Sequence. Fix $0 < r < 1$. Show that if $|x_{n+1} - x_n| < r^n$ for each $n \in \mathbb{N}$, then $\{x_n\}_{n=1}^{\infty}$ converges.

3. § 2.7: Series of Real Numbers. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers and $\sum_{n=1}^{\infty} a_n$ converge. Consider the (decreasing) sequence $\{r_n\}_{n=1}^{\infty}$ of remainders defined by, for $n \in \mathbb{N}$,

$$r_n \ := \ \sum_{i=n}^{\infty} a_i \ .$$

3a. Show that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges.

Hint: First show that for each $N, k \in \mathbb{N}$

$$\sum_{n=N}^{N+k} \frac{a_n}{r_n} \ > \ 1 - \frac{r_{N+k}}{r_N} \ .$$

3b. Show that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$ converges.

Hint: First show that for each $n \in \mathbb{N}$

$$\frac{a_n}{\sqrt{r_n}} \ < \ 2\left(\sqrt{r_n} \ - \ \sqrt{r_{n+1}}\right) \ .$$

4. § 3.1: Open & Closed Sets. Let $\emptyset \neq A \subset \mathbb{R}$. For $c \in \mathbb{R}$, define $\rho(c, A) \in \mathbb{R}$ by

$$\rho(c, A) \ := \ \inf \{ \ |c - a| : a \in A \ \} \ .$$

Intuitively, think of $\rho(c, A)$ as the "distance" between c and A.

4a. Show that x is in the closure of A if and only if $\rho(x, A) = 0$.

Hint. You can use, without proving, a hint to one of the homeworks that says: $x \in \overline{A}$ iff $\exists\ a_n \in A$ so that $\lim_{n\to\infty} a_n = x$.

4b. Show that the set $B := \{x \in \mathbb{R} \colon \rho(x, A) \geq 17\}$ is closed.

5. § 3.2: Compact Sets Let F be a nonempty closed subset of $\mathbb{R}$ and K be a nonempty compact subset of $\mathbb{R}$ such that $K \cap F \ = \ \emptyset$.

5a. Show that there is a $\delta > 0$ such that for each $x \in F$ and $y \in K$ one has that $|x - y| > \delta$.

Hint: Assume there does not exist such a δ and find a contradiction. Recall Theorem 3.2.10: each sequence in a nonempty compact subset K of $\mathbb{R}$ has a subsequence that converges to a point in K.

5b. Show, via an example, that 5a is false if we only assume that K is closed (rather than K is compact).

6. § 4.1: Limit of a Function Let $A = N'_{\varepsilon}(p) := \{x \in \mathbb{R} \colon 0 < |x - p| < \varepsilon\}$ for some $\varepsilon > 0$ and $p \in \mathbb{R}$. Let $f \colon A \to \mathbb{R}$ and $g \colon A \to \mathbb{R}$ be such that

$$\lim_{x\to p} f(x) := \alpha \quad \text{exists} \qquad\qquad \text{and} \qquad\qquad \lim_{x\to p} g(x) := \beta \quad \text{exists} \ .$$

6a. Show that if $f(x) \leq g(x)$ for each $x \in A$ then $\alpha \leq \beta$.

6b. If $f(x) < g(x)$ for each $x \in A$ then is it necessarily true that $\alpha < \beta$? Either prove or give a counterexample.