

# CONVEX FUNCTIONS GIVE INEQUALITIES

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## 1. BASICS

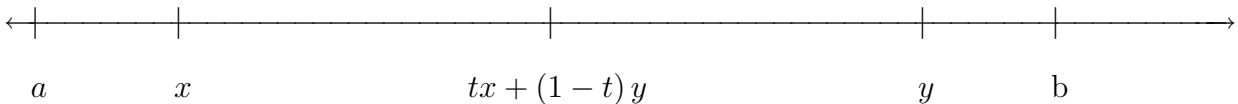
**Notation 1.1.** Throughout this presentation:

- $(a, b) = I \subset \mathbb{R}$  where  $a, b \in \mathbb{R} \cup \{\pm\infty\}$
- $\varphi: I \rightarrow \mathbb{R}$  is a function
- $p \in (1, \infty)$  and its *conjugate exponent*  $p' \in (1, \infty)$  is defined by  $\frac{1}{p} + \frac{1}{p'} = 1$ .

**Definition 1.2.**  $\varphi: I \rightarrow \mathbb{R}$  is *convex*  $\iff$

$$x, y \in I, t \in (0, 1) \implies \varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y). \quad (1.1)$$

**Picture.**



## 2. HOLDER'S INEQUALITY

**Lemma 2.1.** Let  $\varphi: I \rightarrow \mathbb{R}$  be convex. Then:

$$x_i \in I \quad , \quad t_i \in (0, 1) \quad , \quad \sum_{i=1}^n t_i = 1 \quad \implies \quad \varphi \left( \sum_{i=1}^n t_i x_i \right) \leq \sum_{i=1}^n t_i \varphi(x_i) \quad (2.1)$$

$$x_i \in I \quad , \quad t_i \in (0, 1) \quad \implies \quad \varphi \left( \frac{\sum_{i=1}^n t_i x_i}{\sum_{i=1}^n t_i} \right) \leq \frac{\sum_{i=1}^n t_i \varphi(x_i)}{\sum_{i=1}^n t_i} . \quad (2.2)$$

*Proof.* (2.1): Use (1.1) and induction. (2.2): Let  $\tilde{t}_i = [\sum_{j=1}^n t_j]^{-1} t_i$  and apply (2.1). ■

**Recall.** Geometric-Arithmetic Mean Inequality:  $\text{GM} \leq \text{AM}$

$$x_i \geq 0 \quad , \quad n \in \mathbb{N} \quad \implies \quad \left( \prod_{i=1}^n x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n x_i .$$

Using convex functions, we can generalize the GM-AM inequality.

**Proposition 2.2.** Generalized GM-AM inequality:

$$x_i \geq 0 \quad , \quad t_i \in (0, 1) \quad , \quad \sum_{i=1}^n t_i = 1 \quad \implies \quad \prod_{i=1}^n x_i^{t_i} \leq \sum_{i=1}^n t_i x_i . \quad (2.3)$$

*Proof.* Let  $\varphi: (0, \infty) \rightarrow \mathbb{R}$  be  $\varphi(x) = -\ln x$ . Then

$$\begin{aligned} \varphi \text{ is convex} &\implies -\ln \left( \sum_{i=1}^n t_i x_i \right) \leq \sum_{i=1}^n t_i (-\ln x_i) \\ &\implies \sum_{i=1}^n \ln (x_i^{t_i}) \leq \ln \left( \sum_{i=1}^n t_i x_i \right) . \end{aligned} \quad (2.4)$$

Now exponentiate both sides of (2.4). ■

An immediate corollary follows now.

**Corollary 2.3.** Young's Inequality:

$$x_i \geq 0 \quad , \quad 1 < p < \infty \quad \implies \quad x_1 \cdot x_2 \leq \frac{x_1^p}{p} + \frac{x_2^{p'}}{p'} . \quad (2.5)$$

*Proof.* Apply (2.3) to

$$x_1 \cdot x_2 \equiv (x_1^p)^{1/p} \cdot (x_2^{p'})^{1/p'} .$$

■

**Recall.** For a sequence  $\{x_i\}_{i=1}^n = \{x_1, \dots, x_n\}$  from  $\mathbb{R}$ :

$$\|\{x_i\}_{i=1}^n\|_{\ell_p} = \left[ \sum_{i=1}^n |x_i|^p \right]^{1/p} .$$

**Theorem 2.4.** Holder's Inequality in  $\ell_p$ : For sequences  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$  from  $\mathbb{R}$ :

$$\|\{x_i \cdot y_i\}_{i=1}^n\|_{\ell_1} \leq \|\{x_i\}_{i=1}^n\|_{\ell_p} \cdot \|\{y_i\}_{i=1}^n\|_{\ell_{p'}}$$

that is

$$\sum_{i=1}^n |x_i y_i| \leq \left[ \sum_{i=1}^n |x_i|^p \right]^{1/p} \cdot \left[ \sum_{i=1}^n |y_i|^{p'} \right]^{1/p'} . \quad (2.6)$$

When  $p = 2 = p'$ , note that (2.6) is just the Cauchy-Schwarz Inequality.

*Proof.* WLOG: Neither  $\|\{x_i\}_i\|_{\ell_p}$  nor  $\|\{y_i\}_i\|_{\ell_{p'}}$  is 0. WLOG:  $\|\{x_i\}_i\|_{\ell_p} = 1 = \|\{y_i\}_i\|_{\ell_{p'}}$  for if not then:

$$\tilde{x}_i := \frac{x_i}{\|\{x_j\}_j\|_{\ell_p}} \implies \|\{\tilde{x}_i\}_i\|_{\ell_p} = 1 \quad , \quad \tilde{y}_i := \frac{y_i}{\|\{y_j\}_j\|_{\ell_{p'}}} \implies \|\{\tilde{y}_i\}_i\|_{\ell_{p'}} = 1 .$$

By Young's Inequality

$$\sum_{i=1}^n |x_i y_i| \leq \sum_{i=1}^n \left[ \frac{|x_i|^p}{p} + \frac{|y_i|^{p'}}{p'} \right] = \frac{1}{p} \|\{x_i\}_i\|_{\ell_p}^p + \frac{1}{p'} \|\{y_i\}_i\|_{\ell_{p'}}^{p'} = 1 .$$

■

Here are two exercises of Generalized Holder's Inequalities in  $\ell_p$ .

**Exercise.** For sequences  $\{x_i^j\}_{i=1}^n$  from  $\mathbb{R}$  and  $\sum_{j=1}^k \frac{1}{p_j} = 1$

$$\left\| \left\{ \prod_{j=1}^k x_i^j \right\}_{i=1}^n \right\|_{\ell_1} \leq \prod_{j=1}^k \|\{x_i^j\}_{i=1}^n\|_{\ell_{p_j}} .$$

**Exercise.** For sequences  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$  from  $\mathbb{R}$  and  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$

$$\|\{x_i \cdot y_i\}_{i=1}^n\|_{\ell_{p_3}} \leq \|\{x_i\}_{i=1}^n\|_{\ell_{p_1}} \cdot \|\{y_i\}_{i=1}^n\|_{\ell_{p_2}} .$$

Hint:  $\frac{1}{p_1/p_3} + \frac{1}{p_2/p_3} = 1$  .

**Recall.** For a *nice* function  $f: I \rightarrow \mathbb{R}$ ,

$$\|f\|_{L_p} = \left[ \int_I |f(x)|^p dx \right]^{1/p} .$$

**Theorem 2.5.** Holder's Inequality in  $L_p$ : For *nice* functions  $f, g: I \rightarrow \mathbb{R}$ ,

$$\|f \cdot g\|_{L_1} \leq \|f\|_{L_p} \cdot \|g\|_{L_{p'}}$$

that is,

$$\int_I |f(x)g(x)| dx \leq \left[ \int_I |f(x)|^p \right]^{1/p} \cdot \left[ \int_I |g(x)|^{p'} \right]^{1/p'}$$

*Proof.* WLOG: Neither  $\|f\|_{L_p}$  nor  $\|g\|_{L_{p'}}$  is 0 or  $\infty$ . WLOG:  $\|f\|_{L_p} = 1 = \|g\|_{L_{p'}}$  for if not so:

$$\tilde{f} := \frac{f}{\|f\|_{L_p}} \implies \|\tilde{f}\|_{L_p} = 1 \quad , \quad \tilde{g} := \frac{g}{\|g\|_{L_{p'}}} \implies \|\tilde{g}\|_{L_{p'}} = 1 .$$

By Young's Inequality

$$\int_I |f(x)g(x)| dx \leq \int_I \left[ \frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'} \right] = \frac{1}{p} \|f\|_{L_p}^p + \frac{1}{p'} \|g\|_{L_{p'}}^{p'} = 1 .$$

■

Here are two exercises of Generalized Holder's Inequalities in  $L_p$ .

**Exercise.** For *nice* functions  $f_j: I \rightarrow \mathbb{R}$  and  $\sum_{j=1}^k \frac{1}{p_j} = 1$

$$\left\| \prod_{j=1}^k f_j \right\|_{L_1} \leq \prod_{j=1}^k \|f_j\|_{L_{p_j}} .$$

**Exercise.** For *nice* functions  $f, g: I \rightarrow \mathbb{R}$  and  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$

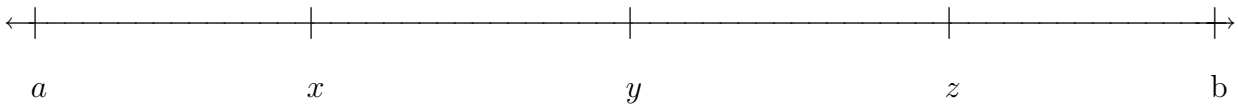
$$\|f \cdot g\|_{L_{p_3}} \leq \|f\|_{L_{p_1}} \cdot \|g\|_{L_{p_2}} .$$

### 3. JENSEN'S INEQUALITY

**Lemma 3.1.** Let  $\varphi: I \rightarrow \mathbb{R}$  be convex. Let  $x, y, z \in I$  with  $x < y < z$ . Then

$$\frac{\varphi(y) - \varphi(x)}{y - x} \leq \frac{\varphi(z) - \varphi(x)}{z - x} \leq \frac{\varphi(z) - \varphi(y)}{z - y}. \quad (3.1)$$

**Picture.**



*Proof.* Key idea:

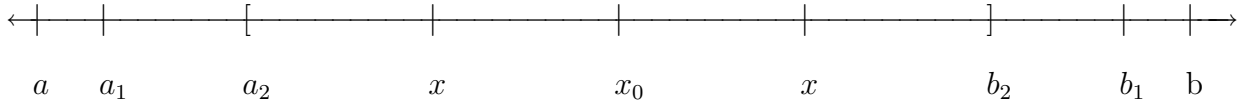
$$y = tx + (1 - t)z = \frac{z - y}{z - x}x + \frac{y - x}{z - x}z \quad \implies \quad \varphi(y) \leq t\varphi(x) + (1 - t)\varphi(z).$$

Think of what this says in the picture; the needed algebra then follows easily. ■

**Proposition 3.2.** Let  $\varphi: I \rightarrow \mathbb{R}$  be convex and  $x_0 \in I$ . Then

- (1)  $\varphi$  is continuous at  $x_0$
- (2)  $\varphi'_-(x_0) := \lim_{x_l \rightarrow x_0^-} \frac{\varphi(x_0) - \varphi(x_l)}{x_0 - x_l}$  exists
- (3)  $\varphi'_+(x_0) := \lim_{x_r \rightarrow x_0^+} \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0}$  exists
- (4)  $\varphi'_-(x_0) \leq \varphi'_+(x_0)$ .

*Proof.* (1): Find:  $a < a_1 < a_2 < x_0 < b_2 < b_1 < b$ . If  $x \in [a_2, b_2] \setminus \{x_0\}$ , then



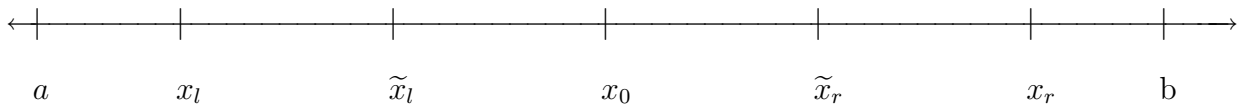
by (3.1)

$$A := \frac{\varphi(a_2) - \varphi(a_1)}{a_2 - a_1} \leq \frac{\varphi(x_0) - \varphi(x)}{x_0 - x} \leq \frac{\varphi(b_1) - \varphi(b_2)}{b_1 - b_2} := B$$

and so

$$\left| \frac{\varphi(x_0) - \varphi(x)}{x_0 - x} \right| \leq \max(|A|, |B|) .$$

(2)–(4): Consider



By (3.1)

$$\frac{\varphi(x_0) - \varphi(x_l)}{x_0 - x_l} \leq \frac{\varphi(x_0) - \varphi(\tilde{x}_l)}{x_0 - \tilde{x}_l} \leq \frac{\varphi(\tilde{x}_r) - \varphi(x_0)}{\tilde{x}_r - x_0} \leq \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0}$$

and so

$$\lim_{x_l \nearrow x_0} \frac{\varphi(x_0) - \varphi(x_l)}{x_0 - x_l} \nearrow \varphi'_-(x_0) \leq \varphi'_+(x_0) \searrow \lim_{x_r \searrow x_0} \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0} . \quad (3.2)$$

■

**Observation 3.3.** Let  $\varphi: I \rightarrow \mathbb{R}$  be convex and  $x_0 \in I$ . Let

$$\varphi'_-(x_0) \leq m \leq \varphi'_+(x_0) .$$

Consider the line

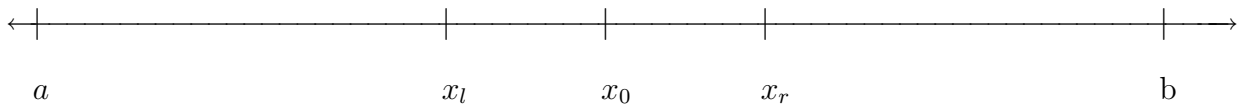
$$l(x) = m(x - x_0) + \varphi(x_0)$$

through the point  $(x_0, \varphi(x_0))$ . Then

$$l(x) \leq \varphi(x) \quad \forall x \in I . \tag{3.3}$$

A line  $y = l(x)$  through the point  $(x_0, \varphi(x_0))$  that satisfies (3.3) is called a *supporting line* of  $y = \varphi(x)$  at  $x_0$ . Draw yourself a picture to see the choice of terminology here. Thus Observation 3.3 says that convex functions always have supporting lines.

**Picture.**



*Proof.* By (3.2)

$$\frac{\varphi(x_0) - \varphi(x_l)}{x_0 - x_l} \leq m \leq \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0} .$$

Thus

$$m(x - x_0) \leq \varphi(x) - \varphi(x_0) \quad \forall x \in I \setminus \{x_0\} .$$

■

**Theorem 3.4.** Jensen's Inequality: Let  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  be convex and  $f: I \rightarrow \mathbb{R}$  be integrable.

If  $I = (0, 1)$ , then

$$\varphi\left(\int_I f(x) dx\right) \leq \int_I \varphi(f(x)) dx \quad (3.4)$$

If  $I = (a, b)$  has finite length, then

$$\varphi\left(\frac{\int_I f(x) dx}{b-a}\right) \leq \frac{\int_I \varphi(f(x)) dx}{b-a}. \quad (3.5)$$

**Remark.** Theorem 3.4 may be thought of as a continuous version of Lemma 2.1.

*Proof.* (3.4): Let  $\int_I f(x) dx = x_0$  and  $\varphi'_-(x_0) \leq m \leq \varphi'_+(x_0)$ . Then by Observation 3.3

$$m(x - x_0) + \varphi(x_0) \leq \varphi(x) \quad \forall x \in \mathbb{R}.$$

Thus

$$m(f(x) - x_0) + \varphi(x_0) \leq \varphi(f(x)) \quad \forall x \in I. \quad (3.6)$$

Now integrate both sides of (3.6) over  $I := (0, 1)$ .

(3.5) follows applying (3.4) to

$$g(x) := f(a + x(b - a)) : I \rightarrow \mathbb{R}.$$

■