

Thm All these approaches to the Riemann Integral are equivalent for bounded functions.  
 " R.I.

Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded.

Then TFAE :

(1)  $f \in \mathcal{R}[a, b]$  in the sense of Def 3.1.3 / tagged partitions, i.e.  
 $(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists \delta > 0) [ \dot{P} \in \dot{\mathcal{P}}_{[a, b]} \text{ and } \|\dot{P}\| < \delta \Rightarrow |S(f, \dot{P}) - L| < \epsilon ]$

(2)  $f$  satisfies the Cauchy Criteria for R.I., i.e.  
 $(\forall \epsilon > 0)(\exists \delta > 0) [ \dot{P}_1, \dot{P}_2 \in \dot{\mathcal{P}}_{[a, b]} \text{ \& } \|\dot{P}_1\| < \delta \text{ \& } \|\dot{P}_2\| < \delta \Rightarrow |S(f, \dot{P}_1) - S(f, \dot{P}_2)| < \epsilon ]$

(3)  $f \in \mathcal{R}[a, b]$  in the sense of upper & lower sums, i.e.

$$\int_a^b f = \bar{\int}_a^b f \quad (:= \tilde{L})$$

side remark Know  $\int_a^b f \leq \bar{\int}_a^b f$

$$\text{so } \int_a^b f = \bar{\int}_a^b f \Leftrightarrow [ (\forall \epsilon > 0) ( \bar{\int}_a^b f - \int_a^b f < \epsilon ) ]$$

(4)  $f$  satisfies Riemann's Integrability Criterion, i.e.

$$(\forall \epsilon > 0) (\exists \mathcal{O} \in \mathcal{P}_{[a, b]}) [ U(\mathcal{O}, f) - L(\mathcal{O}, f) < \epsilon ]$$

Furthermore, if  $f \in \mathcal{R}[a, b]$ , then [the  $L$  from (1) = the  $\tilde{L}$  from (3)]

Pf. Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded, say  $\sup_{x \in [a, b]} |f(x)| := B$ .

$$(2) \begin{matrix} \iff \\ \uparrow \\ \text{already did} \end{matrix} (1) \begin{matrix} \xleftrightarrow{\text{Thm}} \\ 3.2.6 \end{matrix} (3) \begin{matrix} \xleftrightarrow{\text{Thm}} \\ 3.2.7 \end{matrix} (4)$$

Pf... in small steps to come.