CONVEX FUNCTIONS GIVE INEQUALITIES

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1. Basics

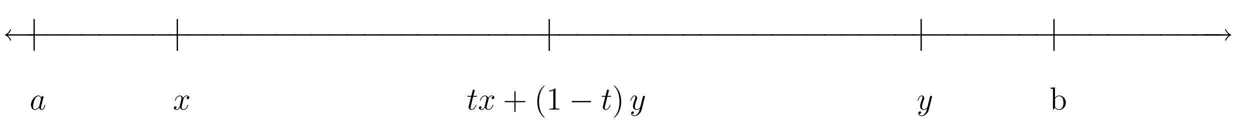
Notation 1.1. Throughout this presentation:

- $(a, b) = I \subset \mathbb{R}$ where $a, b \in \mathbb{R} \cup \{\pm\infty\}$
- $\varphi \colon I \to \mathbb{R}$ is a function
- $p \in (1, \infty)$ and its *conjugate exponent* $p' \in (1, \infty)$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 1.2. $\varphi \colon I \to \mathbb{R}$ is *convex* $\iff$

$$x, y \in I \ , \ t \in (0, 1) \quad \Longrightarrow \quad \varphi\left(tx + (1 - t)\, y\right) \ \leq \ t\varphi(x) + (1 - t)\, \varphi(y) \ . \tag{1.1}$$

Picture.



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2. Holder's Inequality

Lemma 2.1. Let $\varphi\colon I \to \mathbb{R}$ be convex. Then:

$$x_i \in I \quad , \quad t_i \in (0,1) \quad , \quad \sum_{i=1}^{n} t_i = 1 \qquad \Longrightarrow \qquad \varphi\left(\sum_{i=1}^{n} t_i x_i\right) \;\le\; \sum_{i=1}^{n} \; t_i \; \varphi(x_i) \tag{2.1}$$

$$x_i \in I \quad , \quad t_i \in (0,1) \qquad \Longrightarrow \qquad \varphi\left(\frac{\sum_{i=1}^{n} t_i x_i}{\sum_{i=1}^{n} t_i}\right) \;\le\; \frac{\sum_{i=1}^{n} \; t_i \; \varphi(x_i)}{\sum_{i=1}^{n} t_i} \; . \tag{2.2}$$

Proof. (2.1): Use (1.1) and induction. (2.2): Let $\tilde{t}_i = [\sum_{j=1}^{n} t_j]^{-1}\, t_i$ and apply (2.1). ■

Recall. Geometric-Arithmetic Mean Inequality: GM $\le$ AM

$$x_i \ge 0 \quad , \quad n \in \mathbb{N} \qquad \Longrightarrow \qquad \left(\prod_{i=1}^{n} x_i\right)^{1/n} \;\le\; \frac{1}{n}\sum_{i=1}^{n} x_i \; .$$

Using convex functions, we can generalize the GM-AM inequality.

Proposition 2.2. Generalized GM-AM inequality:

$$x_i \ge 0 \quad , \quad t_i \in (0,1) \quad , \quad \sum_{i=1}^{n} t_i = 1 \qquad \Longrightarrow \qquad \prod_{i=1}^{n} x_i^{t_i} \;\le\; \sum_{i=1}^{n} t_i \, x_i \; . \tag{2.3}$$

Proof. Let $\varphi\colon (0,\infty) \to \mathbb{R}$ be $\varphi(x) = -\ln x$. Then

$$\varphi \text{ is convex} \qquad \Longrightarrow \qquad -\ln\left(\sum_{i=1}^{n} t_i \, x_i\right) \;\le\; \sum_{i=1}^{n} t_i \; (-\ln x_i)$$

$$\Longrightarrow \qquad \sum_{i=1}^{n} \ln\left(x_i^{t_i}\right) \;\le\; \ln\left(\sum_{i=1}^{n} t_i \, x_i\right) \; . \tag{2.4}$$

Now exponentiate both sides of (2.4). ■

An immediate corollary follows now.

Corollary 2.3. Young's Inequality:

$$x_i \ge 0 \qquad , \qquad 1 < p < \infty \qquad \Longrightarrow \qquad x_1 \cdot x_2 \;\le\; \frac{x_1^p}{p} + \frac{x_2^{p'}}{p'} \; . \tag{2.5}$$

Proof. Apply (2.3) to

$$x_1 \cdot x_2 \equiv (x_1^p)^{1/p} \cdot \left(x_2^{p'}\right)^{1/p'} .$$

■

Recall. For a sequence $\{x_i\}_{i=1}^n = \{x_1, \ldots, x_n\}$ from $\mathbb{R}$:

$$\|\{x_i\}_{i=1}^n\|_{\ell_p} = \left[\sum_{i=1}^n |x_i|^p\right]^{1/p} .$$

Theorem 2.4. Holder's Inequality in ℓ_p: For sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ from $\mathbb{R}$:

$$\|\{x_i \cdot y_i\}_{i=1}^n\|_{\ell_1} \leq \|\{x_i\}_{i=1}^n\|_{\ell_p} \cdot \|\{y_i\}_{i=1}^n\|_{\ell_{p'}}$$

that is

$$\sum_{i=1}^n |x_i y_i| \leq \left[\sum_{i=1}^n |x_i|^p\right]^{1/p} \cdot \left[\sum_{i=1}^n |y_i|^{p'}\right]^{1/p'} . \tag{2.6}$$

When $p = 2 = p'$, note that (2.6) is just the Cauchy-Schwarz Inequality.

Proof. WLOG: Neither $\|\{x_i\}_i\|_{\ell_p}$ nor $\|\{y_i\}_i\|_{\ell_{p'}}$ is 0. WLOG: $\|\{x_i\}_i\|_{\ell_p} = 1 = \|\{y_i\}_i\|_{\ell_{p'}}$ for if not then:

$$\widetilde{x}_i := \frac{x_i}{\|\{x_j\}_j\|_{\ell_p}} \implies \|\{\widetilde{x}_i\}_i\|_{\ell_p} = 1 \quad , \quad \widetilde{y}_i := \frac{y_i}{\|\{y_j\}_j\|_{\ell_{p'}}} \implies \|\{\widetilde{y}_i\}_i\|_{\ell_{p'}} = 1 .$$

By Young's Inequality

$$\sum_{i=1}^n |x_i y_i| \leq \sum_{i=1}^n \left[\frac{|x_i|^p}{p} + \frac{|y_i|^{p'}}{p'}\right] = \frac{1}{p} \|\{x_i\}_i\|_{\ell_p}^p + \frac{1}{p'} \|\{y_i\}_i\|_{\ell_{p'}}^{p'} = 1 .$$

■

Here are two exercises of Generalized Holder's Inequalities in ℓ_p.

Exercise. For sequences $\{x_i^j\}_{i=1}^n$ from $\mathbb{R}$ and $\sum_{j=1}^k \frac{1}{p_j} = 1$

$$\left\| \left\{ \prod_{j=1}^k x_i^j \right\}_{i=1}^n \right\|_{\ell_1} \leq \prod_{j=1}^k \|\{x_i^j\}_{i=1}^n\|_{\ell_{p_j}} .$$

Exercise. For sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ from $\mathbb{R}$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$

$$\|\{x_i \cdot y_i\}_{i=1}^n\|_{\ell_{p_3}} \ \le \ \|\{x_i\}_{i=1}^n\|_{\ell_{p_1}} \ \cdot \ \|\{y_i\}_{i=1}^n\|_{\ell_{p_2}} \ .$$

Hint: $\frac{1}{p_1/p_3} + \frac{1}{p_2/p_3} = 1$.

Recall. For a *nice* function $f\colon I \to \mathbb{R}$,

$$\|f\|_{L_p} \ = \ \left[\int_I |f(x)|^p \ dx\right]^{1/p} \ .$$

Theorem 2.5. Holder's Inequality in L_p: For *nice* functions $f, g\colon I \to \mathbb{R}$,

$$\|f \cdot g\|_{L_1} \ \le \ \|f\|_{L_p} \ \cdot \ \|g\|_{L_{p'}}$$

that is,

$$\int_I |f(x)g(x)| \ dx \ \le \ \left[\int_I |f(x)|^p\right]^{1/p} \ \cdot \ \left[\int_I |g(x)|^{p'}\right]^{1/p'}$$

Proof. WLOG: Neither $\|f\|_{L_p}$ nor $\|g\|_{L_{p'}}$ is 0 or ∞. WLOG: $\|f\|_{L_p} \ = \ 1 \ = \ \|g\|_{L_{p'}}$ for if not so:

$$\widetilde{f} := \frac{f}{\|f\|_{L_p}} \ \Longrightarrow \ \left\|\widetilde{f}\right\|_{L_p} = 1 \qquad , \qquad \widetilde{g} := \frac{g}{\|g\|_{L_{p'}}} \ \Longrightarrow \ \|\widetilde{g}\|_{L_{p'}} = 1 \ .$$

By Young's Inequality

$$\int_I |f(x)g(x)| \ dx \ \le \ \int_I \left[\frac{|f(x)|^p}{p} \ + \ \frac{|g(x)|^{p'}}{p'}\right] \ = \ \frac{1}{p} \ \|f\|_{L_p}^p \ + \ \frac{1}{p'} \ \|g\|_{L_{p'}}^{p'} \ = \ 1 \ .$$

■

Here are two exercises of Generalized Holder's Inequalities in L_p.

Exercise. For *nice* functions $f_j\colon I \to \mathbb{R}$ and $\sum_{j=1}^k \frac{1}{p_j} = 1$

$$\left\|\prod_{j=1}^k f_j\right\|_{L_1} \ \le \ \prod_{j=1}^k \|f_j\|_{L_{p_j}} \ .$$

Exercise. For *nice* functions $f, g\colon I \to \mathbb{R}$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$

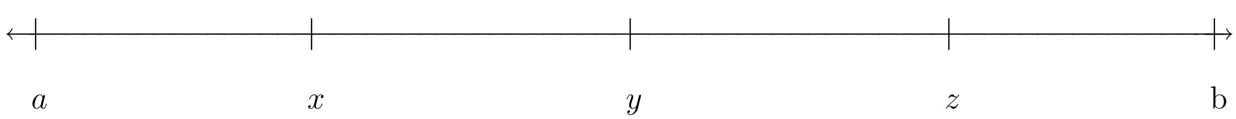
$$\|f \cdot g\|_{L_{p_3}} \ \le \ \|f\|_{L_{p_1}} \ \cdot \ \|g\|_{L_{p_2}} \ .$$

3. Jensen's Inequality

Lemma 3.1. Let $\varphi\colon I \to \mathbb{R}$ be convex. Let $x, y, z \in I$ with $x < y < z$. Then

$$\frac{\varphi(y)-\varphi(x)}{y-x} \;\le\; \frac{\varphi(z)-\varphi(x)}{z-x} \;\le\; \frac{\varphi(z)-\varphi(y)}{z-y}\,. \tag{3.1}$$

Picture.



Proof. Key idea:

$$y = tx + (1-t)z = \frac{z-y}{z-x}\,x + \frac{y-x}{z-x}\,z \qquad \Longrightarrow \qquad \varphi(y) \;\le\; t\varphi(x) + (1-t)\varphi(z)\,.$$

Think of what this says in the picture; the needed algebra then follows easily. ∎

Proposition 3.2. Let $\varphi \colon I \to \mathbb{R}$ be convex and $x_0 \in I$. Then

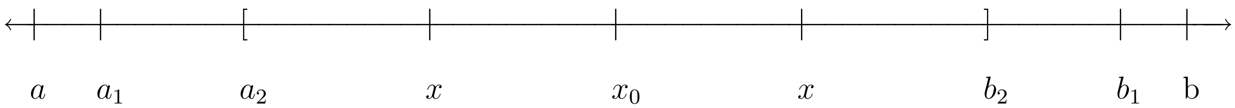
(1) φ is continuous at x_0

(2) $\varphi'_-(x_0) := \lim_{x_l \to x_0^-} \frac{\varphi(x_0)-\varphi(x_l)}{x_0 - x_l}$ exists

(3) $\varphi'_+(x_0) := \lim_{x_r \to x_0^+} \frac{\varphi(x_r)-\varphi(x_0)}{x_r - x_0}$ exists

(4) $\varphi'_-(x_0) \leq \varphi'_+(x_0)$.

Proof. (1): Find: $a < a_1 < a_2 < x_0 < b_2 < b_1 < b$. If $x \in [a_2, b_2] \setminus \{x_0\}$, then



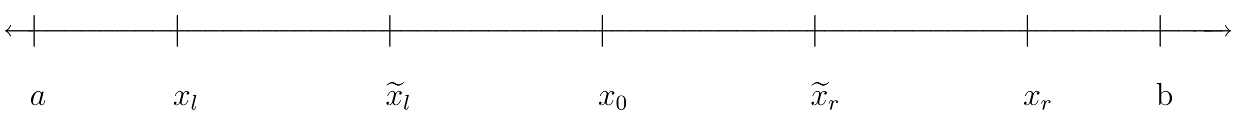
by (3.1)

$$A \;:=\; \frac{\varphi(a_2)-\varphi(a_1)}{a_2 - a_1} \;\leq\; \frac{\varphi(x_0)-\varphi(x)}{x_0 - x} \;\leq\; \frac{\varphi(b_1)-\varphi(b_2)}{b_1 - b_2} \;:=\; B$$

and so

$$\left| \frac{\varphi(x_0)-\varphi(x)}{x_0 - x} \right| \;\leq\; \max\left(|A|, |B|\right) \;.$$

(2)–(4): Consider



By (3.1)

$$\frac{\varphi(x_0)-\varphi(x_l)}{x_0 - x_l} \;\leq\; \frac{\varphi(x_0)-\varphi(\widetilde{x}_l)}{x_0 - \widetilde{x}_l} \;\leq\; \frac{\varphi(\widetilde{x}_r)-\varphi(x_0)}{\widetilde{x}_r - x_0} \;\leq\; \frac{\varphi(x_r)-\varphi(x_0)}{x_r - x_0}$$

and so

$$\lim_{x_l \nearrow x_0} \frac{\varphi(x_0)-\varphi(x_l)}{x_0 - x_l} \;\nearrow\; \varphi'_-(x_0) \;\leq\; \varphi'_+(x_0) \;\swarrow\; \lim_{x_r \searrow x_0} \frac{\varphi(x_r)-\varphi(x_0)}{x_r - x_0} \;. \qquad (3.2)$$

■

Observation 3.3. Let $\varphi\colon I \to \mathbb{R}$ be convex and $x_0 \in I$. Let

$$\varphi'_-(x_0) \ \le \ m \ \le \ \varphi'_+(x_0) \ .$$

Consider the line

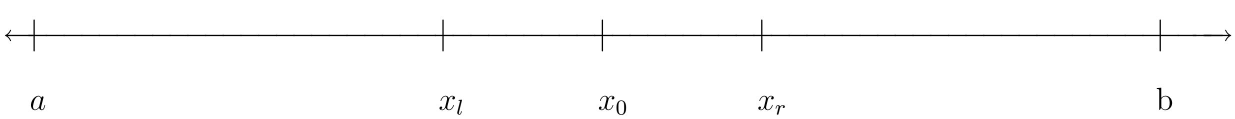
$$l(x) = m(x - x_0) + \varphi(x_0)$$

through the point $(x_0, \varphi\,(x_0))$. Then

$$l(x) \ \le \ \varphi(x) \qquad\qquad \forall x \in I \ . \tag{3.3}$$

A line $y = l(x)$ through the point $(x_0, \varphi\,(x_0))$ that satisfies (3.3) is called a *supporting line* of $y = \varphi(x)$ at x_0. Draw yourself a picture to see the choice of terminolgy here. Thus Observation 3.3 says that convex functions always have supporting lines.

Picture.



Proof. By (3.2)

$$\frac{\varphi(x_0) - \varphi(x_l)}{x_0 - x_l} \ \le \ m \ \le \ \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0} \ .$$

Thus

$$m(x - x_0) \ \le \ \varphi(x) - \varphi(x_0) \qquad\qquad \forall x \in I \setminus \{x_0\} \ .$$

■

Theorem 3.4. Jensen's Inequality: Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be convex and $f\colon I \to \mathbb{R}$ be integrable. If $I = (0,1)$, then

$$\varphi\left(\int_I f(x)\,dx\right) \;\le\; \int_I \varphi\left(f\left(x\right)\right)\,dx \tag{3.4}$$

If $I = (a,b)$ has finite length, then

$$\varphi\left(\frac{\int_I f(x)\,dx}{b-a}\right) \;\le\; \frac{\int_I \varphi\left(f\left(x\right)\right)\,dx}{b-a}\;. \tag{3.5}$$

Remark. Theorem 3.4 may be thought of as a continuous version of Lemma 2.1.

Proof. (3.4): Let $\int_I f(x)\,dx = x_0$ and $\varphi'_-(x_0) \;\le\; m \;\le\; \varphi'_+(x_0)$. Then by Observation 3.3

$$m(x-x_0) + \varphi(x_0) \;\le\; \varphi(x) \qquad\qquad \forall x \in \mathbb{R}\;.$$

Thus

$$m(f(x)-x_0) + \varphi(x_0) \;\le\; \varphi\left(f\left(x\right)\right) \qquad\qquad \forall x \in I\;. \tag{3.6}$$

Now integrate both sides of (3.6) over $I := (0,1)$.

(3.5) follows applying (3.4) to

$$g(x) \;:=\; f\left(a + x\left(b-a\right)\right) : I \to \mathbb{R}\;.$$

■