

Ideas/Defs

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in \mathbb{R} . Define two corresponding sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$, which examine the behavior of the *tails of* $\{s_n\}_{n=1}^{\infty}$, by

$$\begin{aligned} a_k &= \inf_{n \geq k} s_n &= \text{glb}\{s_n : n \geq k\} &\in \mathbb{R} \cup \{-\infty\} && \text{an increasing } \nearrow \text{ sequence} \\ b_k &= \sup_{n \geq k} s_n &= \text{lub}\{s_n : n \geq k\} &\in \mathbb{R} \cup \{\infty\} && \text{an decreasing } \searrow \text{ sequence.} \end{aligned}$$

Since $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are monotone sequences, their limits exist in the extended ($\widehat{\mathbb{R}}$) sense. Now define (and examine) the *limit superior* and *limit inferior* of the sequence $\{s_n\}_{n=1}^{\infty}$.

$$\begin{aligned} \liminf_{n \rightarrow \infty} s_n &\stackrel{\text{notation}}{=} \underline{\lim}_{n \rightarrow \infty} s_n \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \inf_{n \geq k} s_n = \lim_{k \rightarrow \infty} a_k \stackrel{\text{Thm 4.1.6}}{=} \sup_{k \in \mathbb{N}} a_k \in \widehat{\mathbb{R}} \\ \limsup_{n \rightarrow \infty} s_n &\stackrel{\text{notation}}{=} \overline{\lim}_{n \rightarrow \infty} s_n \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \sup_{n \geq k} s_n = \lim_{k \rightarrow \infty} b_k \stackrel{\text{Thm 4.1.6}}{=} \inf_{k \in \mathbb{N}} b_k \in \widehat{\mathbb{R}} \end{aligned}$$

Claim 1. $\underline{\lim}_{n \rightarrow \infty} s_n \leq \overline{\lim}_{n \rightarrow \infty} s_n$

Claim 2. $\{s_n\}_n$ is bounded below $\Leftrightarrow \underline{\lim}_{n \rightarrow \infty} s_n \in \mathbb{R}$. $\{s_n\}_n$ is bounded above $\Leftrightarrow \overline{\lim}_{n \rightarrow \infty} s_n \in \mathbb{R}$.

Claim 3. Let $\overline{\lim}_{n \rightarrow \infty} s_n \in \mathbb{R}$.

$$\overline{\lim}_{n \rightarrow \infty} s_n \leq \beta \iff \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad , \quad s_n < \beta + \varepsilon$$

Claim 4. Let $\overline{\lim}_{n \rightarrow \infty} s_n \in \mathbb{R}$.

$$\overline{\lim}_{n \rightarrow \infty} s_n \geq \beta \iff \forall \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad , \quad \beta - \varepsilon < s_n$$

Claim 5. Let $\underline{\lim}_{n \rightarrow \infty} s_n \in \mathbb{R}$.

$$\underline{\lim}_{n \rightarrow \infty} s_n \geq \alpha \iff \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad , \quad \alpha - \varepsilon < s_n$$

Claim 6. Let $\underline{\lim}_{n \rightarrow \infty} s_n \in \mathbb{R}$.

$$\underline{\lim}_{n \rightarrow \infty} s_n \leq \alpha \iff \forall \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad , \quad s_n < \alpha + \varepsilon$$

Claim 7. $\underline{\lim}_{n \rightarrow \infty} s_n = -\overline{\lim}_{n \rightarrow \infty} (-s_n)$

Claim 8. $\lim_{n \rightarrow \infty} s_n$ exists in the extended sense (i.e. in $\widehat{\mathbb{R}}$) $\iff \underline{\lim}_{n \rightarrow \infty} s_n = \overline{\lim}_{n \rightarrow \infty} s_n$.

In this case, $\underline{\lim}_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} s_n = \overline{\lim}_{n \rightarrow \infty} s_n$.

Corollary. Let $\overline{\lim}_{n \rightarrow \infty} s_n \in \mathbb{R}$. Then $\overline{\lim}_{n \rightarrow \infty} s_n = \beta$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad , \quad s_n < \beta + \varepsilon \quad (3)$$

and

$$\forall \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad , \quad \beta - \varepsilon < s_n \quad (4)$$

Corollary. Let $\underline{\lim}_{n \rightarrow \infty} s_n \in \mathbb{R}$. Then $\underline{\lim}_{n \rightarrow \infty} s_n = \alpha$ if and only if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad , \quad \alpha - \varepsilon < s_n \quad (5)$$

and

$$\forall \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad , \quad s_n < \alpha + \varepsilon \quad (6)$$

Claim 9.

- (1) $\overline{\lim}_{n \rightarrow \infty} s_n = \infty \Leftrightarrow \forall M \in \mathbb{R} \quad \forall N \in \mathbb{N} \quad \exists n \geq N \text{ s.t. } s_n \geq M$
- (2) $\underline{\lim}_{n \rightarrow \infty} s_n = -\infty \Leftrightarrow \forall M \in \mathbb{R} \quad \forall N \in \mathbb{N} \quad \exists n \geq N \text{ s.t. } s_n \leq M$
- (3) $\overline{\lim}_{n \rightarrow \infty} s_n = -\infty \Leftrightarrow \lim_{n \rightarrow \infty} s_n = -\infty$
- (4) $\underline{\lim}_{n \rightarrow \infty} s_n = \infty \Leftrightarrow \lim_{n \rightarrow \infty} s_n = \infty$

Claim 10. There exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ of $\{s_n\}_n$ s.t. $\lim_{k \rightarrow \infty} s_{n_k} = \overline{\lim}_{n \rightarrow \infty} s_n \in \widehat{\mathbb{R}}$.

Also there exists a subsequence $\{s_{n_j}\}_{j=1}^{\infty}$ of $\{s_n\}_n$ s.t. $\lim_{j \rightarrow \infty} s_{n_j} = \underline{\lim}_{n \rightarrow \infty} s_n \in \widehat{\mathbb{R}}$.

Freebie: each bounded sequence has a convergent subsequence.

Claim 11. Let $\{s_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{s_n\}_n$ such that $\lim_{k \rightarrow \infty} s_{n_k} \in \widehat{\mathbb{R}}$.

Then $\underline{\lim}_{n \rightarrow \infty} s_n \leq \lim_{k \rightarrow \infty} s_{n_k} \leq \overline{\lim}_{n \rightarrow \infty} s_n$.

Claim 12.

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} s_n &= \sup \left\{ \lim_{k \rightarrow \infty} s_{n_k} : \{s_{n_k}\}_{k=1}^{\infty} \text{ is a subsequence of } \{s_n\}_n \text{ s.t. } \lim_{k \rightarrow \infty} s_{n_k} \in \widehat{\mathbb{R}} \right\} \\ \underline{\lim}_{n \rightarrow \infty} s_n &= \inf \left\{ \lim_{k \rightarrow \infty} s_{n_k} : \{s_{n_k}\}_{k=1}^{\infty} \text{ is a subsequence of } \{s_n\}_n \text{ s.t. } \lim_{k \rightarrow \infty} s_{n_k} \in \widehat{\mathbb{R}} \right\} \end{aligned}$$

Claim 13. provided it makes sense:

$$\underline{\lim}_{n \rightarrow \infty} s_n + \underline{\lim}_{n \rightarrow \infty} t_n \stackrel{(2)}{\leq} \underline{\lim}_{n \rightarrow \infty} (s_n + t_n) \stackrel{(\text{Claim 1})}{\leq} \overline{\lim}_{n \rightarrow \infty} (s_n + t_n) \stackrel{(1)}{\leq} \overline{\lim}_{n \rightarrow \infty} s_n + \overline{\lim}_{n \rightarrow \infty} t_n$$

even more

$$\underline{\lim}_{n \rightarrow \infty} s_n + \underline{\lim}_{n \rightarrow \infty} t_n \stackrel{(2)}{\leq} \underline{\lim}_{n \rightarrow \infty} (s_n + t_n) \stackrel{(4)}{\leq} \underline{\lim}_{n \rightarrow \infty} s_n + \overline{\lim}_{n \rightarrow \infty} t_n \stackrel{(3)}{\leq} \overline{\lim}_{n \rightarrow \infty} (s_n + t_n) \stackrel{(1)}{\leq} \overline{\lim}_{n \rightarrow \infty} s_n + \overline{\lim}_{n \rightarrow \infty} t_n$$