

MARK BOX		
PROBLEM	POINTS	
1	10	
2	10	
3	10	
4	10	
5	10	
bonus	1	
Total	30	
%	100	

Math 554/703i.002      Prof. Girardi

Fall 98      Final Exam      12/08/98

NAME: \_\_\_\_\_

**INSTRUCTIONS:**

- Write a NEAT FORMAL proof to **3** of the **5** problems.  
I am doing problem numbers: \_\_\_\_\_ .
- Use your own paper:
  - write on only one side of the page
  - begin each problem on a new page
  - put your name on each page.
- The MARK BOX indicates the problems along with their points.  
Check that your copy of the exam has all of the problems.
- During this test, do not leave your seat.  
If you have a question, raise your hand.
- This closed book/notes exam covers (*Intro. to Real Analysis*, 1<sup>st</sup> ed., by Stoll):  
Sections 1.1 – 1.7, 2.1 – 2.7, 3.1 – 3.2 .

**Problem Source:**

- 1998 Exam 1 # 5
- Similar to 1998 Exam 2 # 1
- 1998 Exam 3 # 3
- 1998 Exam 3 # 4
- Theorem 3.1.6 from the text

**Bonus:** What is your favorite compact subset of  $\mathbb{R}$ ?

1. Let  $A$  and  $B$  be arbitrary sets with  $A_1 \subset A$  and  $B_1 \subset B$ .

Consider a function  $f: A \rightarrow B$ .

- 1a.  $f$  is one-to-one provided that: if  $a_1, a_2 \in A$  and  $a_1 \neq a_2$ , then \_\_\_\_\_ .  
1b. The definition of  $f^{-1}(B_1)$  says that:  $a \in f^{-1}(B_1)$  if and only if \_\_\_\_\_ .  
1c. Show that  $A_1 \subset f^{-1}f(A_1)$ .  
1d. Show that  $f$  is one-to-one if and only if  $D = f^{-1}f(D)$  for all subsets  $D$  of  $A$ .
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2. Countability

- 2a. Let  $\mathfrak{M}$  be an *infinite* set. Let  $C$  be any countable set.

If there exists a function  $f: \mathfrak{M} \rightarrow C$  such that  $f$  is \_\_\_\_\_, then  $\mathfrak{M}$  is countable.

If there exists a function  $f: C \rightarrow \mathfrak{M}$  such that  $f$  is \_\_\_\_\_, then  $\mathfrak{M}$  is countable.

Since  $\mathbb{Q}$  is countable,  $\mathbb{Q} \times \mathbb{Q}$  is \_\_\_\_\_.

A countable union of countable sets is \_\_\_\_\_.

- 2b. Let  $\mathfrak{M}$  be the set of all finite subsets of  $\mathbb{Q}$  i.e.,

$$\mathfrak{M} = \{M: M \subset \mathbb{Q} \text{ and } M \text{ has only a finite number of elements} \} .$$

Show that  $\mathfrak{M}$  is countable.

HINT: You may use any fact from **2a** without proving it.

HINT: Consider, for  $n \in \mathbb{N}$ ,

$$\mathfrak{M}_n = \{M: M \subset \mathbb{Q} \text{ and } M \text{ has } n \text{ elements} \} .$$

WARNING: Watch your notation.

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3. Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $\mathbb{R}$ . Let  $\{\varepsilon_n\}$  be a sequence of POSITIVE real numbers satisfying

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0 .$$

- 3a. By definition,  $\{x_n\}$  is CAUCHY if

\_\_\_\_\_ .

- 3b. A BIG-TIME theorem says that  $\{x_n\}$  is Cauchy if and only if

\_\_\_\_\_ .

- 3c. Let  $\{x_n\}$  be Cauchy and furthermore, for each  $n \in \mathbb{N}$ ,

$$|x_n - y_n| < \varepsilon_n .$$

Show that  $\{y_n\}$  is Cauchy. You may use **3a** but you may NOT use **3b**.

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4. Let  $\{x_n\}$  be a bounded sequence in  $\mathbb{R}$ .
- 4a. By definition,  $\overline{\lim}_{n \rightarrow \infty} x_n = \lim_{k \rightarrow \infty} b_k$  where  $b_k = \sup\{\underline{\hspace{2cm}}\}$ .
- 4b. Clearly show that there exists a subsequence  $\{x_{n_k}\}_{k=1}^{\infty}$  of  $\{x_n\}_{n=1}^{\infty}$  satisfying

$$\lim_{k \rightarrow \infty} x_{n_k} = \overline{\lim}_{n \rightarrow \infty} x_n .$$

HINT: Use the definition from 4a to construct such a subsequence.

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5. Let  $\{O_i : i \in \mathbb{N}\}$  be a collection of open subsets of  $\mathbb{R}$ .
- 5a. A set  $O \subset \mathbb{R}$  is *open* if and only if  $\forall p \in O \exists \varepsilon > 0$  such that  $\underline{\hspace{2cm}}$ .
- 5b. Using the definition from 5a., show that  $\bigcap_{i=1}^n O_i$  is open.
- 5c. Is  $\bigcap_{i=1}^{\infty} O_i$  necessarily open? Prove so or give a specific counterexample.