MARK BOX		
PROBLEM	POINTS	
1	10	
2	10	
3	10	
4	10	
Total	30	
%	100	

Math 554/703i.002 Prof. Girardi

Fall 98 Exam 2 10/29/98

NAME: ______________________________

INSTRUCTIONS:

1. Write a NEAT FORMAL proof to **3** of the **4** problems.
   I am doing problem numbers: ____ .
2. Use your own paper:
   a. write on only one side of the page
   b. begin each problem on a new page
   c. put your name on each page.
2. The MARK BOX indicates the problems along with their points.
   Check that your copy of the exam has all of the problems.
3. During this test, do not leave your seat.
   If you have a question, raise your hand.
4. This closed book/notes exam covers (*Intro. to Real Analysis*, 1st ed., by Stoll):
   Sections 1.7, 2.1 – 2.3.

Problem Source:

1. an easier version of look at problem § 1.7 # 15
2. look at problem § 2.1 # 12
3. hand in problem § 2.2 # 12
4. hand in problem § 2.3 # 13

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1. Countability

1a. Let I be an *infinite* set.

If there exists a function $f\colon I \to \mathbb{N}$ such that f is ____________ , then I is countable.

If there exists a function $f\colon \mathbb{N} \to I$ such that f is ____________ , then I is countable.

Since $\mathbb{Q}$ is countable, $\mathbb{Q} \times \mathbb{Q}$ is ____________________.

1b. Let I be the set of all closed intervals with rational endpoints, i.e.,

$$I \;=\; \{[a,b] : a \in \mathbb{Q} \quad \text{and} \quad b \in \mathbb{Q} \quad \text{and} \quad a < b\} \;.$$

Show that I is countable.

HINT: You may use any fact from **1a** without proving it.

WARNING: Watch your notation.

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2. Convergent Sequences: let $\{x_n\}_{n=1}^{\infty}$ be sequence of real numbers and $x \in \mathbb{R}$.

2a. By definition, $\lim_{n\to\infty} x_n = x$ if and only if

for each $\varepsilon > 0$ ____ .

2b. Let $\lim_{n\to\infty} x_n = x$. Show that $\lim_{n\to\infty} x_n^2 = x^2$.

HINT: $x_n^2 - x^2 \;=\; (x_n + x)\;(x_n - x)\;.$

2c. Let $\lim_{n\to\infty} x_n^2 = x^2$. Is it necessarily true that $\lim_{n\to\infty} x_n = x$?

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3. Convergent Sequences: let $\{x_n\}_{n=1}^{\infty}$ be sequence of real numbers and $x \in \mathbb{R}$ and for each $n \in \mathbb{N}$, let

$$s_n \;=\; \frac{x_1 + x_2 + \ldots + x_n}{n} \;.$$

3a. By definition, $\lim_{n\to\infty} x_n = x$ if and only if

for each $\varepsilon > 0$ ____ .

3b. Let $\lim_{n\to\infty} x_n = x$. Show that $\lim_{n\to\infty} s_n = x$.

3c. Give an example of a sequence $\{x_n\}$ which diverges but for which $\{s_n\}$ converges.

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4. Divergence to infinity: let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of real numbers.

4a. By definition, $\lim_{n\to\infty} x_n = \infty$ if and only if

____ .

4b. Let $\lim_{n\to\infty} x_n = \infty$ amd $\{y_n\}$ is a convergent sequence.

Show that $\lim_{n\to\infty}(x_n + y_n) = \infty$