

MARK BOX		
PROBLEM	POINTS	
1	10	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total	100	

Math 554/703i.002 12/10/96 Final Exam

NAME: _____

SSN: _____

INSTRUCTIONS:

- (1) On the PROOF PROBLEMS, write only a NEAT FORMAL proof/definition on the page. If needed, do your THINKING SCRATCH WORK/SKELETON on the BACK of the PREVIOUS page. Failure to follow this may result in no points.
- (2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
- (3) This is a closed book/closed notes exam covering (from *An Introduction to Analysis* by M. Stoll – 1996 preprint edition) § 1.1 – 1.7, 2.1 – 2.6, 3.1 – 3.2, 4.1 – 4.2.

NOTATION: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3 \dots\}$, \mathbb{Q} is the set of rational numbers, and \mathbb{R} is the set of real numbers. All other notation is as it was in class.

1. If the statement is true, then circle T. If the statement is false, then circle F. The **NEW** scoring is 1 points for a correct answer, 1/2 point for a blank answer, and 0 points for an incorrect answer. In this problem, unless otherwise stated, $A_i \subset A$ and $B_i \subset B$ and f is a function from A to B .

T or F : [1] $(A_1 \cup A_2)^C = A_1^C \cap A_2^C$.

T or F : [2] $(A_1 \cup A_2)^C = A_1^C \cup A_2^C$.

T or F : [3] $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

T or F : [4] $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

T or F : [5] \mathbb{Z} is countable.

T or F : [6] \mathbb{R} is countable.

T or F : [7] A bounded sequence $\{x_n\}$ in \mathbb{R} converges if and only if the $\liminf x_n$ and $\limsup x_n$ both exist.

T or F : [8] A bounded sequence $\{x_n\}$ in \mathbb{R} converges if and only if $\liminf x_n = \limsup x_n$.

T or F : [9] A point $p \in \mathbb{R}$ is a limit point of $E \subset \mathbb{R}$ if, for each $\varepsilon > 0$, we have that $N'_\varepsilon(p) \cap E \neq \emptyset$. Remember: $N'_\varepsilon(p) = N_\varepsilon(p) \setminus \{p\}$.

T or F : [10] A point $p \in E$ is an isolated point of $E \subset \mathbb{R}$ if, for each $\varepsilon > 0$, we have that $N'_\varepsilon(p) \cap E = \emptyset$.

2. Let A , B , and C be subsets of some big set X . Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) .$$

REMARK: This is Theorem 1.1.2 from the text ... please give a prove of it.

3a. By DEFINITION, a sequence $\{p_n\}$ in \mathbb{R} converges to $p \in \mathbb{R}$ if:

for all $\varepsilon > 0$ _____ .

When this is the case, we write $\lim_{n \rightarrow \infty} p_n = p$.

3b. Rigorously show, using the definition in **3a**, that $\lim_{n \rightarrow \infty} \frac{2n + 3}{n + 1} = 2$.

4. Complete the table below indicating:

- (1) $\text{Int}(E)$, i.e. the set of interior points of E ,
- (2) the isolated points of E ,
- (3) whether E is open, closed, both, or neither.

An answer of “open” in the last column will mean that you think E is “open and not closed”. An answer of “closed” in the last column will mean that you think E is “closed and not open”. You do **not** need to show work on this problem.

E	$\text{Int}(E)$	isolated pts. of E	open/closed?
$(2, 3)$			
$(0, \infty) \cap \mathbb{Q}$			
$[0, \infty)$			
$\{2/n : n \in \mathbb{N}\}$			
\emptyset			

5. Elephant Hunting.
- 5a. The **Nested Intervals Property** says that if $\{I_n\}_{n=1}^\infty$ is a sequence of nested intervals (ie. $I_n \supset I_{n+1}$ for each $n \in \mathbb{N}$) and each I_n is _____ and _____, then $\bigcap_{n=1}^\infty I_n \neq \emptyset$.
- 5b. By **definition**, a subset K of \mathbb{R} is compact if and only if each open covering of K has a finite subcovering. To phrase this differently, K is compact if and only if: for each collection $\{U_\alpha\}_{\alpha \in A}$ of open sets satisfying

$$K \subset \bigcup_{\alpha \in A} U_\alpha$$

there exists a finite number of the α 's, say $\alpha_1, \alpha_2, \dots, \alpha_n$, such that

$$\text{_____} \subset \bigcup_{i=1}^n U_{\alpha_i} .$$

- 5c. The Heine-Borel Theorem states that each closed bounded interval $[a, b]$ in \mathbb{R} is compact. Finish the below proof of the Heine-Borel Theorem.

Proof. Fix an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of $[a, b]$. Thus, each U_α is open and $[a, b] \subset \bigcup_{\alpha \in A} U_\alpha$. Assume that:

$$\mathcal{U} \text{ has no finite subcover of } [a, b] . \quad (*)$$

(We want to get a contraction).

Divide $[a, b]$ into two closed subintervals

$$\left[a, \frac{a+b}{2} \right] \quad \text{and} \quad \left[\frac{a+b}{2}, b \right]$$

each of length $\frac{(b-a)}{2}$. By (*), at least one of these, call it I_1 , cannot be covered by a finite number of the U_α 's. Repeat this process to obtain a sequence $\{I_n\}_{n=1}^\infty$ of closed and bounded intervals satisfying

- (1) $[a, b] \supset I_1 \supset I_2 \supset I_3 \dots$,
- (2) the length of I_n is $|I_n| = \frac{(b-a)}{2^n}$,
- (3) for each $n \in \mathbb{N}$, I_n cannot be covered by a finite number of the U_α 's.

By the nested interval property, $\bigcap_{n=1}^\infty I_n \neq \emptyset$. So there exists $x \in \bigcap_{n=1}^\infty I_n$.

Thus

$$x \in \bigcap_{n=1}^\infty I_n \subset [a, b] \subset \bigcup_{\alpha \in A} U_\alpha .$$

So there exists $\alpha_o \in A$ such that $x \in U_{\alpha_o}$.

Continue the proof on the next (blank) page.

Space to continue with the proof of the Heine-Borel Theorem:

6. Prove the following (big-time) Theorem:

Let K be a compact subset of \mathbb{R} and $f: K \rightarrow \mathbb{R}$ be a continuous function. Set

$$M = \max\{f(x) : x \in K\} .$$

Then there exists $b \in K$ such that $M = f(b)$.

You may use, without proving, the following theorems from class.

(A) Theorem Let $x_n, x \in K \subset \mathbb{R}$. Let $f: K \rightarrow \mathbb{R}$ be a continuous function.

If $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$.

(HB)=(B) Heine-Borel Thm – Thm 3.2.9 : For $E \subset \mathbb{R}$:

E is compact $\iff E$ is closed and bounded.

(BW)=(C) Bolzano-Weierstrass Thm – Cor 2.4.11 :

Each bounded sequence in \mathbb{R} has a convergent subsequence.

(D) Thm. 3.1.11 : For $E \subset \mathbb{R}$:

E is closed $\iff E$ contains all its limit point $\iff E = E \cup E'$.

(E) Thm 2.4.7: If q is a limit point of $E \subset \mathbb{R}$, then there exists a sequence $\{q_n\}$ in E such that $q_n \rightarrow q$.

(F) Thm 2.4.3: If $a_n \rightarrow a \in \widehat{\mathbb{R}}$, then each subsequence of $\{a_n\}$ also converges to a .

(G) Thm 2.1.11: If a sequence $\{p_n\}$ converges, then its limit is unique.

(H) Corollary: If $\{x_n\}$ is a sequence in a compact subset K of \mathbb{R} , then $\{x_n\}$ has a subsequence that converges to some point in K .

7. Do one (the choice is yours), and only one, of the following 3 problems. Fill in the blank and then do the proof on the next (blank) page.

7a. Let $\{x_n\}$ be a bounded sequence in \mathbb{R} and let $\{y_n\}$ be another sequence with the following property:

- (1) for each y_n there is a subsequence of $\{x_n\}$ that converges to y_n
- (2) $\{y_n\}$ converges, say $\lim_{n \rightarrow \infty} y_n = y$.

Show that there exists a subsequence of $\{x_n\}$ that converges to y .

Hint: So you need to construct a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}$ that converges to y .

By definition, the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to y if and only if

$\forall \varepsilon > 0 \exists k_\varepsilon \in \mathbb{N}$ such that _____ .
(yes ... it is k_ε and not n_ε).

7b. Let A and B be two open subsets of \mathbb{R} and define

$$A + B = \{a + b \in \mathbb{R} : a \in A \text{ and } b \in B\} .$$

Show that $A + B$ is also open.

Hint: By definition $A + B$ is open if and only if for each $p \in A + B$

there exists $\varepsilon > 0$ such that _____ .

So start by fixing $p \in A + B$. Then you know that there exists $a \in A$ and $b \in B$ such that $p = a + b$

7c. Let $M > 0$. Let $f: [a, b] \rightarrow \mathbb{R}$ be a function that satisfies

$$|f(x) - f(y)| \leq M |x - y| ,$$

for each $x, y \in [a, b]$. Show that f is continuous on $[a, b]$.

Hint: By definition, f is continuous on $[a, b]$ if and only if

$\forall x \in [a, b]$ and $\forall \varepsilon > 0 \exists$ _____ such that

if $y \in N_\delta(x) \cap [a, b]$ then $f(y) \in N_\varepsilon(f(x))$.

Below, I am doing problem 7 ____ .