| MARK BOX | | |
|---|---|---|
| PROBLEM | POINTS | |
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |

Math 554/703i.002 9/26/96 Exam 1

NAME: ______________________________

SSN: ________________________________

**INSTRUCTIONS**:

(1) On the PROOF PROBLEMS, write only a NEAT FORMAL proof/definition on the page. If needed, do your THINKING SCRATCH WORK/SKELETON on the BACK of the PREVIOUS page. Failure to follow this may result in no points.
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) This is a closed book/closed notes exam covering (from *An Introduction to Analysis* by M. Stoll – 1996 preprint edition) Chapter 1.

**1.** If the statement is true, then circle T. If the statement is false, then circle F. The scoring is 2 points for a correct answer, 1 point for a blank answer, and 0 points for an incorrect answer. In this problem, unless otherwise stated, $A_i \subset A$ and $B_i \subset B$ and $f$ is a function from $A$ to $B$.

T or F : 1) $(A_1 \cup A_2)^C = A_1^C \cup A_2^C$ .

T or F : 2) $(A_1 \cup A_2)^C = A_1^C \cap A_2^C$ .

T or F : 3) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

T or F : 4) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

T or F : 5) $a$ is the least upper bound of $A \subset \mathbb{R}$ if
(i) $a$ is an upper bound of $A$ and
(ii) if $b \in \mathbb{R}$ and $b$ is an upper bound of $A$, then $a \leqslant b$.

T or F : 6) $a$ is the least upper bound of $A \subset \mathbb{R}$ if
(i) $a$ is an upper bound of $A$ and
(ii) if $b \in \mathbb{R}$ and $b$ is an upper bound of $A$, then $b \leqslant a$.

T or F : 7) Every nonempty bounded subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$ .

T or F : 8) Every nonempty bounded subset of $\mathbb{Q}$ has a least upper bound in $\mathbb{Q}$ .

T or F : 9) $\mathbb{N} \sim \mathbb{Z}$, i.e., $\mathbb{Z}$ is countable.

T or F : 10) $\mathbb{N} \sim \mathbb{R}$, i.e., $\mathbb{R}$ is countable.

**2.** Let $A$, $B$, and $C$ be subsets of some big set $X$. Prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) .$$

REMARK: This is Theorem 1.1.2b and problem 6a from § 1.1 from the text.

**3.** Consider a function $f\colon \{0, 1, 2, \dots\} \to \mathbb{R}$ that is defined recursively by $f(0) = 5$ and $f(1) = 1$ and

$$f(n+1) = f(n) + 2f(n-1)$$

and for $n \geq 1$. Use math induction to show that

$$f(n) = 2^{n+1} + 3(-1)^n$$

for all integers $n \geq 0$.

**4.** Countability.

**4a.** Fill in the blanks. Your choice of words is:

1-to-1, onto, finite, infinite .

A set $A$ is countable if one of the below (equivalent) properties holds:

(1) there is a map $f\colon \mathbb{N} \to A$ that is ______________ and ______________ .

(2) there is a map $f\colon A \to \mathbb{N}$ that is ______________ and ______________ .

(3) there is a map $f\colon \mathbb{N} \to A$ where $f$ is __________ and $A$ is ___________ .

(4) there is a map $f\colon A \to \mathbb{N}$ where $f$ is __________ and $f(A)$ is __________ .

**4b.** Let $B$ be an infinite set. Show that $B$ contains a countable set.

REMARK: This is problem 19 from § 1.7.

HINT: Use (3) above.

**5.** Let $A$ and $B$ be arbitrary sets with $A_1 \subset A$ and $B_1 \subset B$. Consider a function $f\colon A \to B$.

**5a.** $f$ is one-to-one provided that: if $a_1, a_2 \in A$ and $a_1 \neq a_2$, then ________________ .

**5b.** The definition of $f^{-1}(B_1)$ says that: $a \in f^{-1}(B_1)$ if and only if ___________ .

**5c.** Show that $A_1 \subset f^{-1}f(A_1)$. [hint: think of $f(A_1)$ as $B_1$.]

**5d.** Show that $f$ is one-to-one if and only if $D = f^{-1}f(D)$ *for all* subsets $D$ of $A$.