MARK BOX		
Problem	Points	
1	30	
2	30	
3	30	
4 i	25	
4 ii	25	
Total	140	

MATH 554 FALL 1993 FINAL EXAM

NAME: ____________________

SSN: ____________________

Instructions:

(1) *On the "proof problems", write only a* <u>*neat*</u> *formal proof (and definition) on the page. If so needed, do your "thinking scratch work" on the* <u>*back*</u> *of the* <u>*previous*</u> *page. Failure to follow this may result in no points.*
(2) The "Mark Box" indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) This is a closed book/closed notes exam covering (from *An Introduction to Analysis* by M. Stoll – 1993 preprint version) § 1.1–1.4, 1.6, 2.1–2.6, 3.1–3.7, 4.1–4.3 & 5.1–5.2.
(4) Part 3 consists of 3A & 3B & 3C. Do 1 of the 3 problems. Put a **BIG X** through the pages of the problems which you do NOT want us to count.
(5) Part 4 consists of 4A & 4B & 4C. Do 2 of the 3 problems. Put a **BIG X** through the page of the problem which you do NOT want us to count. In Part 4, you may use, without proving, any result from class.

Throughout this exam, the notation is:

(X, d) denotes an arbitrary metric space.
E denotes a subset of X.
$\boxed{\{p_n\} \text{ is a sequence in } E}$ (i. e. $p_n \in E$).

$\mathbb{R}$ denotes the set of real numbers.
$\mathbb{Q}$ denotes the set of rational numbers.
$\boxed{\{a_n\}, \{b_n\} \;\&\; \{c_n\} \text{ denote sequences in } \mathbb{R}}$.

Other notation is as in class.

1. If the statement is true, then circle T. If the statement is false, then circle F. The scoring is 2 points for a correct answer, 1 point for a blank answer, and 0 points for a incorrect answer.

1) T or F : Every nonempty subset of $\mathbb{Q}$ which is bounded above has a least upper bound in $\mathbb{Q}$.

2) T or F : Every closed set in $\mathbb{R}$ is compact.

3) T or F : If the sequence $\{a_n\}$ is not bounded above, then $\lim_{n\to\infty} a_n = \infty$.

4) T or F : Every bounded sequence in $\mathbb{R}$ is convergent.

5) T or F : Every convergent sequence is bounded.

6) T or F : For any sequence $\{c_n\}$ in $\mathbb{R}$, the corresponding sequence $\{b_k\}$ where $b_k = \sup\{c_n : n \geq k\}$ is monotone increasing.

7) T or F : Every convergent sequence $\{p_n\}$ is a Cauchy sequence.

8) T or F : If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

9) T or F : For a function $f\colon X \to \mathbb{R}$ and $p \in X$, if $\lim_{x\to p} f(x)$ exists then f is continuous at p.

10) T or F : If the function $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

11) T or F : The function $f(x) = \frac{1}{x}$ is uniformly continuous on $[1,2]$.

12) T or F : The function $f(x) = x^2$ is uniformly continuous on $\mathbb{R}$.

13) T or F : If the function $f\colon X \to \mathbb{R}$ is continuous on $\mathbb{R}$, then the set $f^{-1}(\{0\})$ is closed.

14) T or F : If the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, then f is differentiable.

15) T or F : If the functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are both differentiable on $\mathbb{R}$, then the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by $h(x) = g(f(x))$ is differentiable and $h'(x) = g'(x) \cdot f'(x)$.

** 2 ** 2 ** 2 ** 2 ** 2 **

Consider a function $f\colon [a,b] \to \mathbb{R}$. Fix $p \in (a,b) \subset \mathbb{R}$.

2-1) By a remark in the book, f is continuous at p if and only if $\lim_{x\to p} f(x) =$ ____________________ .

2-2) By definition, f is differentiable at p if the following limit exists:

____ .

2-3) Reproduce the proof of the following theorem:
If f is differentiable at p, then f is continuous at p.

** 3A ** 3A ** 3A ** 3A ** 3A **

Let A be a nonempty subset of $\mathbb{R}$ which is bounded above.

3A-1) By definition, $\alpha \in \mathbb{R}$ is the **least upper bound** of A, i.e. $\alpha = \text{l.u.b. } A$, if:

(i) α is ____
and
(ii) if $\beta < \alpha$, then ____ .

3A-2) By definition, α is a **limit point** of *the set* A, if:

$\forall \epsilon > 0$ ____ .

3A-3) Reproduce the proof of the following homework problem:
Let A be a nonempty subset of $\mathbb{R}$ which is bounded above and let $\alpha = \text{l.u.b. } A$.
Show that if α is not in A, then α is a limit point of A.

** 3B ** 3B ** 3B ** 3B ** 3B **

Fix $E \subset \mathbb{R}$. Consider functions $f\colon E \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$. Fix $p \in E$.

3B-1) By definition, f is bounded on E if

$\underline{\exists M}$ ____ .

3B-2) By definition, $\lim_{x\to p} f(x) = L$ if

$\underline{\forall \epsilon > 0}$ ____ .

3B-3) Reproduce the proof of the following homework problem:
If f is bounded on E and $\lim_{x\to p} g(x) = 0$, prove (ϵ, δ) that $\lim_{x\to p} f(x)\, g(x) = 0$.

** 3C ** 3C ** 3C ** 3C ** 3C **

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) \quad = \quad \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} .$$

3C-2) Using the definition of differentiable (see your (2-2)), show that f is differentiable at $x = 0$ and that $f'(0) = 0$.

3C-3) Show that $f'(x)$ is NOT continuous at $x = 0$.
You may use elementary calculus (eg. $D_t \sin t = \cos t$) to compute $f'(x)$ for $x \neq 0$.

** 4A ** 4A ** 4A ** 4A ** 4A **

Let A and B be arbitrary sets and consider a function $f\colon A \to B$.

4A-1) By definition, for f to be one-to-one means that if $a_1, a_2 \in A$ and $a_1 \neq a_2$ then

____ .

4A-2) Show that if $C \subset A$ then $C \subset f^{-1}f(C)$.

4A-3) Show that f is one-to-one if and only if $C = f^{-1}f(C)$ *for all* subsets C of A.

** 4B ** 4B ** 4B ** 4B ** 4B **

4B-1) Theorem 3.8 form the book states that:
Every sequence in a compact set K has a convergent ______________________ .

(Hint) For the next part, you may use the following observation without proving it.
Let $c \in E \subset X$ and let $f\colon E \to R$ be a continuous function. Then if a sequence $\{x_n\}$ in E converges to some point $c \in E$, then the sequence $\{f(x_n)\}$ coverges to $f(c)$.

4B-2) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function satisfying that for each $a \in [0,1]$ there is some $b \in [0,1]$ such that

$$(*) \qquad |f(b)| \leq \frac{1}{2}|f(a)| \ .$$

Show that there is a $c \in [0,1]$ such that $f(c) = 0$.

⊛ Use the condition $(*)$ to construct a sequence $\{x_n\}$ of points in $[0,1]$ such that $|f(x_{n+1})| \leq \frac{1}{2}\,|f(x_n)|$ and then compare $|f(x_n)|$ with $|f(x_1)|$

** 4C ** 4C ** 4C ** 4C ** 4C **

4C-1) By definition, a function $f\colon X \to \mathbb{R}$ is *uniformly continuous* on X if:

$\underline{\forall \epsilon > 0}$ ____ .

4C-2) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ which is differentiable on (a,b). The *Mean Value Theorem* states that there exists an $c \in (a,b)$ such that:

____ .

4C-3) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and $f'(x)$ is bounded. Prove that f is uniformly continous on $\mathbb{R}$.