MARK BOX		
Problem	Points	
1	20	
2	20	
3 i	20	
4 i	20	
4 ii	20	
Total	100	

MATH 554 FALL 1993 EXAM # 3

NAME: ______________________________

SSN: ______________________________

Instructions:

(1) *On the "proof problems", write only a* <u>*neat*</u> *formal proof (and definition) on the page. If so needed, do your "thinking scratch work" on the* <u>*back*</u> *of the* <u>*previous*</u> *page. Failure to follow this may result in no points.*
(2) The "Mark Box" indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) This is a closed book/closed notes exam covering (from *An Introduction to Analysis* by M. Stoll – 1993 preprint version) § 3.3, 3.4, 3.5, 3.6, 3.7
(4) Throughout this exam, the notation is:
(X, d) denotes an arbitrary metric space. E denotes a subset of X.
$\boxed{\{p_n\} \text{ is a sequence in } E}$ (i. e. $p_n \in E$).
$\mathbb{R}$ denotes the set of real numbers. $\boxed{\{a_n\},\ \{b_n\}\ \&\ \{c_n\} \text{ denote sequences in } \mathbb{R}}$.
Other notation is as in class.
(5) Part 3 consists of 3A & 3B. Do 1 of the 2 problems. Put a **BIG X** through the page of the problem which you do NOT want us to count.
(6) Part 4 consists of 4A & 4B & 4C. Do 2 of the 3 problems. Put a **BIG X** through the page of the problem which you do NOT want us to count. In Part 4, you may use, without proving, any result from class.

1. If the statement is true, then circle T. If the statement is false, then circle F. The scoring is 2 points for a correct answer, 1 point for a blank answer, and 0 points for a incorrect answer.

1) T or F : Every bounded sequence in $\mathbb{R}$ is convergent.
2) T or F : If the sequence $\{a_n\}$ is not bounded above, then $\lim_{n\to\infty} a_n = \infty$.
3) T or F : If p is a limit point of E, then there is a seq. in E which converges to p.
4) T or F : If the sequence $\{p_n\}$ has a subsequence which converges, then the sequence $\{p_n\}$ also converges.
5) T or F : A bounded sequence $\{c_n\}$ is convergent if and only if $\overline{\lim}\, c_n = \underline{\lim}\, c_n$.
6) T or F : $\underline{\lim}(a_n + b_n) \ \geq\ \underline{\lim}\, a_n + \underline{\lim}\, b_n$
7) T or F : Every convergent sequence $\{p_n\}$ is a Cauchy sequence.
8) T or F : If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.
9) T or F : For any sequence $\{c_n\}$ in $\mathbb{R}$, the corresponding sequence $\{a_k\}$ where $a_k = \inf\{c_n : n \geq k\}$ is monotone increasing.
10) T or F : For any sequence $\{c_n\}$ in $\mathbb{R}$, the corresponding sequence $\{b_k\}$ where $b_k = \sup\{c_n : n \geq k\}$ is monotone increasing.

** 2 ** 2 ** 2 ** 2 ** 2 ** 2 **

2-1) By definition, the sequence $\{a_n\}$ in $\mathbb{R}$ converges to $a \in \mathbb{R}$ if:

$\underline{\forall \epsilon > 0}$ ____ .

2-2) Reproduce the proof of the following theorem:
If $\{a_n\}$ is monotone and bounded, then $\{a_n\}$ converges.

** 3A ** 3A ** 3A ** 3A ** 3A **

Let $\{a_n\}$ and $\{b_n\}$ be *bounded* sequences in $\mathbb{R}$. Show that

$$\overline{\lim}(a_n + b_n) \ \le \ \overline{\lim}\, a_n + \overline{\lim}\, b_n \ .$$

** 3B ** 3B ** 3B ** 3B ** 3B **

Let $s_1 = 0$. For $n \geq 1$, let s_n be defined by

$$s_{2m} \;=\; \tfrac{1}{2}\, s_{_{2m-1}} \qquad \text{and} \qquad s_{2m+1} \;=\; \tfrac{1}{2} \;+\; s_{2m}\ .$$

Show that $\overline{\lim}_{n\to\infty} s_n = 1$ and $\underline{\lim}_{n\to\infty} s_n = \frac{1}{2}$.

** 4A ** 4A ** 4A ** 4A ** 4A **

4A-1) By definition, the sequence $\{a_n\}$ in $\mathbb{R}$ is *monotone increasing* if:

____ .

4A-2) Let $\{a_n\}_{n=1}^{\infty}$ be a monotone increasing sequence in $\mathbb{R}$. Prove that if a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converges, then the sequence $\{a_n\}_{n=1}^{\infty}$ also converges.

$\circledast$ Hint: In problem 2, you wrote out the definition of convergent sequence.

** 4B ** 4B ** 4B ** 4B ** 4B **

4B-1) Construct an example of bounded sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}$ such that $\overline{\lim}(a_n + b_n) \neq \overline{\lim}\, a_n + \overline{\lim}\, b_n$. Give a *brief* explanation of your example.

4B-2) Construct an example of bounded sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}$ such that $\overline{\lim}(a_n \cdot b_n) \neq \overline{\lim}\, a_n \cdot \overline{\lim}\, b_n$. Give a *brief* explanation of your example.

** 4C ** 4C ** 4C ** 4C ** 4C **

Recall that by definition, a sequence $\{a_n\}$ in $\mathbb{R}$ is *Cauchy* if:

$$\forall\, \epsilon > 0 \quad \exists\, N_0 \in I \qquad \text{such that} \qquad \forall\, n, m \geq N_0 \quad |a_n - a_m| < \epsilon\ .$$

4C-1) Prove that if the sequence $\{a_n\}$ in $\mathbb{R}$ is Cauchy, then for any real number λ, the sequence $\{\lambda a_n\}$ is also Cauchy.

4C-2) Prove that if the sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}$ are both Cauchy, then the sequence $\{a_n + b_n\}$ is also Cauchy.