MARK BOX		
Problem	Points	
1	20	
2	20	
3 i	20	
4 i	20	
4 ii	20	
Total	100	

MATH 554 FALL 1993 EXAM # 2

NAME: ______________________________

SSN: ______________________________

Instructions:

(1) *On the "proof problems", write only a* <u>*neat*</u> *formal proof (and definition) on the page. If so needed, do your "thinking scratch work" on the* <u>*back*</u> *of the* <u>*previous*</u> *page. Failure to follow this may result in no points.*
(2) The "Mark Box" indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) This is a closed book/closed notes exam covering (from *An Introduction to Analysis* by M. Stoll – 1993 preprint version) sections 2.1–2.6 & 3.1–3.2
(4) Throughout this exam, the notation is:
$\mathbb{R}$ denotes the set of real numbers.
(X, d) denotes an arbitrary metric space. A , B & E denote subsets of X .
$\{a_n\}$, $\{b_n\}$ & $\{c_n\}$ denote sequences in $\mathbb{R}$.
Other notation is as in class.
(5) Part 3 consists of 3A & 3B. Do 1 of the 2 problems. Put a **BIG X** through the page of the problem which you do NOT want us to count.
(6) Part 4 consists of 4A & 4B & 4C. Do 2 of the 3 problems. Put a **BIG X** through the page of the problem which you do NOT want us to count. In Part 4, you may use, without proving, any result from class.

1. If the statement is true, then circle T. If the statement is false, then circle F. The scoring is 2 points for a correct answer, 1 point for a blank answer, and 0 points for a incorrect answer.

1) T or F : If p is a limit point of E , then every ϵ-neighborhood $N_\epsilon(p)$ of p contains infinitely many points of E .

2) T or F : The intersection of any collection of open sets is open.

3) T or F : The intersection of any collection of closed sets is closed.

4) T or F : Every closed set is compact.

5) T or F : If a subset of $\mathbb{R}$ is bounded, then it is compact.

6) T or F : The empty set is both open and closed.

7) T or F : A set E is closed if and only if $E = \overline{E}$.

8) T or F : Every convergent sequence is bounded.

9) T or F : If $\lim_{n\to\infty} a_n = 0$, then $\{a_n\ b_n\}$ converges and $\lim_{n\to\infty} a_n\ b_n = 0$.

10) T or F : For any real number p , $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

** 2 ** 2 ** 2 ** 2 ** 2 ** 2 **

2-1) By definition, a point $p \in E$ is an **interior point** of E if:

____ .

2-2) By definition, E is **open** if each point in E is:

____ .

2-3) It follows from a theorem in class that E is closed if and only if its complement E^c is: ____________________________ .

2-4) Reproduce the proof of the following theorem: Every compact set is closed.

** 3A ** 3A ** 3A ** 3A ** 3A **

Let $\{G_\alpha\}$ be any collection of open subsets of X . Using the definition of open set (see your part 2), show that $\cup_{\alpha \in A} G_\alpha$ is open.
Remark: Do NOT use deMorgan's Law – use only the definition of open.

** 3B ** 3B ** 3B ** 3B ** 3B **

3B-1) By definition, $\lim_{n\to\infty} a_n = a$ if for each $\epsilon > 0$:

____ .

3B-2) Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Show that $\lim_{n\to\infty}(a_n + b_n) = a + b$.

** 4A ** 4A ** 4A ** 4A ** 4A **

4A-1) Show that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

4A-2) Show that if $B \equiv \cup_{i=1}^{\infty} A_i$, then $\cup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{B}$. Hint: the previous part is helpful.

4A-3) Give an example where equality does NOT hold in (4A-2). No proof needed.

** 4B ** 4B ** 4B ** 4B ** 4B **

4B-1) By definition, A is compact if every open covering of A has a

____ .

4B-2) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be a function with the property that:

$(\ast)$ if O is an open subset in Y, then $f^{-1}(O)$ is an open subset in X.

Let K be a compact subset of X. **Using the definition** from (4B-1) and **property** $(\ast)$, show that $f(K)$ is a compact subset of Y. Recall from Chapter 1 (and use without proving) that

$$K \subset f^{-1}f(K) \qquad \text{and} \qquad f^{-1}\left(\bigcup_{\alpha\in A} O_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(O_\alpha)$$

$$f\ f^{-1}(O) \subset O \qquad \text{and} \qquad f\left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n} f(B_i)\ .$$

** 4C ** 4C ** 4C ** 4C ** 4C **

4C-1) By definition, $\lim_{n\to\infty} b_n = b$ if for each $\epsilon > 0$:

____ .

4C-2) Prove the following squeeze law. Suppose that for each $n \in \mathbb{N}$, $a_n \le b_n \le c_n$. Also suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$. Show that $\lim_{n\to\infty} b_n = L$.