MARK BOX		
Problem	Points	
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

MATH 554 FALL 1993 EXAM # 1

NAME: ____________________

SSN: ____________________

Instructions:

(1) *On the "proof problems", write only a* <u>*neat*</u> *formal proof (and definition) on the page. If so needed, do your "thinking scratch work" on the* <u>*back*</u> *of the* <u>*previous*</u> *page. Failure to follow this may result in no points.*
(2) The "Mark Box" indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) This is a closed book/closed notes exam covering (from *An Introduction to Analysis* by M. Stoll – 1993 preprint version) Chapter 1.
(4) $\mathbb{R}$ denotes the set of real numbers. $\mathbb{Q}$ denotes the set of rational numbers.

1. If the statement is true, then circle T. If the statement is false, then circle F. The scoring is 2 points for a correct answer, 1 point for a blank answer, and 0 points for a incorrect answer. In this problem, unless otherwise stated, $A_i \subset A$ and $B_i \subset B$ and f is a function from A to B.

1) T or F : $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

2) T or F : $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

3) T or F : The function f is one-to-one if it assigns to each element $x \in A$ a unique element $y \in B$.

4) T or F : $(0, \infty) \sim [0, 1]$, i.e., $(0, \infty)$ is equivalent to $[0, 1]$.

5) T or F : Every nonempty subset of $\mathbb{Q}$ which is bounded above has a least upper bound in $\mathbb{Q}$.

6) T or F : If $x, y \in \mathbb{R}$ and $x < y$, then there exists $r \in \mathbb{Q}$ such that $x < r < y$.

7) T or F : If A and B are nonempty subsets of $\mathbb{R}$ which are bounded above, then l.u.b. $(A + B) =$ l.u.b. $A +$ l.u.b. B.

8) T or F : If A and B are nonempty subsets of $\mathbb{R}$ which are bounded above, then l.u.b. $(AB) =$ (l.u.b. A)(l.u.b. B).

9) T or F : The set of all finite (i.e. terminating) sequences of 0's and 1's is countable.

10) T or F : The set of *all* sequences of 0's and 1's is countable.

2. Consequence of the Least Upper Bound Property (LUB Property).

   a) The Least Upper Bound Property for $\mathbb{R}$ states that:

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      ____ .

   b) Reproduce the proof of the following Theorem from the text:
      For any real numbers x and y, with $x > 0$, there is a positive integer n such that

$$nx > y \; .$$

3. Least Upper Bound. Let A be a nonempty subset of $\mathbb{R}$ which is bounded above.

a) By definition, $\alpha \in \mathbb{R}$ is the **least upper bound** of A, i.e. $\alpha = \text{l.u.b. } A$, if:

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b) Reproduce the proof of the following homework problem:
Let $\alpha = \text{l.u.b. } A$. Show that for any $\epsilon > 0$, there is some $x \in A$ such that

$$\alpha - \epsilon < x \leq \alpha.$$

4. Countability For 4b&c, you may use (without proving) theorems and corollaries which we covered in class. Let $I = \{1, 2, \dots\}$ denotes the set of positive integers.

a) By definition, an infinite set A is countable if

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b) Use the fact that $[0, 1]$ is uncountable to show that $\mathbb{R}$ is uncountable.

c) Show that the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

5. Let $\{C_n\}$ be sequence of non-trivial closed bounded intervals of $\mathbb{R}$, say $C_n = [a_n, b_n]$. Furthermore, let $\{C_n\}$ be a **nested** sequence, i.e.

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \cdots \qquad .$$

Draw yourself a picture to see that this implies that if $k < n$, then $C_n \subset C_k$ and so

$$a_k \leq a_n < b_n \leq b_k \ .$$

Show that there exists a number $x \in \mathbb{R}$ such that $x \in C_n$ for each $n \in I$.