MARK BOX		
Problem	Points	You
1	13	
2	13	
3	13	
4	14	
Total	40	

MATH 554 Spring 1992 FINAL EXAM
NAME: ______________________________

SSN: ______________________________

Instructions:

(1) *Write only a neat formal proof on the page.* If you need more space, continue on the back of the previous page. Do your "thinking scratch work" on scratch paper. Of course, I will not look at your scratch work.
(2) Do 3 of the 4 problems. Do # 3 & #4. Do either # 1 or #2.
(3) WARNING: I will be very particular in the grading. Take the time to organize your thoughts and then *thoroughly* explain your idea. *Basically, convince me that you understand.* Use correct English. You have plenty of time.
(4) Textbook: *An Introduction to Analysis*, 1^{st} ed., by James R. Kirkwood.

1. Let $f\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}$ be continuous functions on $[0,1]$ that satisfy

(*) $$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx \ .$$

Show that there is a point $c \in [0,1]$ such that $f(c) = g(c)$.

⊛ This is a (very short quick but kinda clever and definitely neat) application of the Mean Value Theorem for Integrals (page 150). But be careful.

2. Let $f\colon (0,1) \to \mathbb{R}$ be a differentiable function satisfying for each $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - y| < \delta$ and $x, y \in (0,1)$ then

(*) $$\left|\frac{f(x) - f(y)}{x - y} - f'(x)\right| < \epsilon \ .$$

Show that f' is *uniformly continuous* on $(0,1)$.

⊛ This is a "fix $\epsilon > 0$ - find δ that works" proof – straight from the definition of uniform continuity. The proof is short. The triangle inequality is helpful.

⊛ Be very specific in your explanation. Say exactly what it means for "δ to work".

Do either #1 or #2 on the next page.

I am doing # ____________.

3. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function satisfying that for each $a \in [0,1]$ there is a $b \in [0,1]$ such that

$$(*) \qquad |f(b)| \leq \frac{1}{2}|f(a)| \ .$$

Show that there is a $c \in [0,1]$ such that $f(c) = 0$.

⊛ Use the condition $(*)$ to construct a sequence $\{x_n\}$ of points in $[0,1]$ such that $|f(x_{n+1})| \leq \frac{1}{2}|f(x_n)|$.

⊛ How do $|f(x_n)|$ and $|f(x_1)|$ compare?

⊛ What can you know say about $\lim_{n\to\infty} f(x_n)$?

⊛ Now to find c. ... Enough hints ...

4. Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function and let $\alpha\colon [0,1] \to \mathbb{R}$ be a non-decreasing function. Let f be Reimann-Stieltjes integrable with respect to α, ie. $f \in \mathcal{R}(\alpha)$. Also, there is a *positive* number K such that $K < f(x)$ for all $x \in [0,1]$. Use theorem 6-22 (ie. the " the work-horse") to show that the function $\frac{1}{f}$ is Reimann-Stieltjes integrable with respect to α.

⊛ You may use the notation from class without explaining it.

⊛ Fix $\epsilon > 0$ and find a partition P that works. But tell me what it means "to work".

⊛ Helpful is to compare $M_i(\frac{1}{f})$ and $m_i(\frac{1}{f})$ and $M_i(f)$ and $m_i(f)$. No formal proof of how these comapre is necessary; just make the observation.