MARK BOX		
Problem	Points	You
1	50	
2	10	
3	20	
4	20	
Total	100	

MATH 554 Spring 1992 EXAM 2

NAME: ______________________________

SSN: ______________________________

Instructions:

(1) *On the "proof problems", write only a neat formal proof (and definition) on the page. If so needed, do your "thinking scratch work" on the back of the previous page. Failure to follow this may result in no points.*
(2) This is a closed book/closed notes exam covering sections 2.3, 3.1, and 4.1.
(3) During this exam, do not leave your seat. Raise your hand if you have a question. When you finish, turn your exam over, put your pencil down, and raise your hand.
(4) The "Mark Box" indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(5) $\mathbb{R}$ denotes the set of real numbers.
$\{x_n\}$ and $\{y_n\}$ are sequences.
L is a real number.
A is a subset of $\mathbb{R}$.
(6) X and Y are subsets of $\mathbb{R}$ and f is a function from X into Y. So $f : X \to Y$.
(7) Throughout this exam, convergence is understood to mean convergence in the finite sense (to some finite real number and not to $+\infty$ nor $-\infty$).
(8) Textbook: *An Introduction to Analysis*, 1^{st} ed., by James R. Kirkwood.

1. If the statement is true, then circle T. If the statement if false, then circle F. The scoring is 2 points for a correct answer, 1 point for a blank answer, and 0 points for a incorrect answer.

⊛ section 2.3 – Bolzano Weierstrass Theorem:

T or F : 1) Every bounded infinite set of real numbers has at least one limit point.

T or F : 2) For a bounded sequence $\{x_n\}$, the $\limsup x_n = L$ if and only if for each $\epsilon > 0$ there are infinitely many terms of $\{x_n\}$ in $(L-\epsilon, L+\epsilon)$ but only finitely many terms of $\{x_n\}$ with $x_n < L - e$.

T or F : 3) For a bounded sequence $\{x_n\}$, the $\liminf x_n = L$ if and only if for each $\epsilon > 0$ there are infinitely many terms of $\{x_n\}$ in $(L-\epsilon, L+\epsilon)$. but only finitely many terms of $\{x_n\}$ with $x_n < L - \epsilon$

T or F : 4) $\{x_n\}$ converges if and only if it is bounded and has exactly one subsequential limit point.

T or F : 5) Every bounded sequence has a convergent subsequence.

T or F : 6) Every sequence has a convergent subsequence.

T or F : 7) A bounded sequence $\{x_n\}$ converges if and only if the $\liminf x_n$ and $\limsup x_n$ both exist.

T or F : 8) A bounded sequence $\{x_n\}$ converges if and only if $\liminf x_n = \limsup x_n$.

T or F : 9) For bounded sequences $\{x_n\}$ and $\{y_n\}$, $\limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n$.

T or F : 10) For bounded sequences $\{x_n\}$ and $\{y_n\}$, $\limsup (x_n + y_n) \geq \limsup x_n + \limsup y_n$.

⊛ section 3.1 – topology

T or F : 11) The arbitrary intersection of open sets is an open set.

T or F : 12) The arbitrary intersection of closed sets is a closed set.

T or F : 13) $x \in A$ is an *interior point* of A if for all $\delta > 0$ we have that $(x - \delta, x + \delta) \subset A$.

T or F : 14) $x \in A$ is a *boundary point* of A if for each $\delta > 0$ the interval $(x - \delta, x + \delta)$ contains a point in A <u>and</u> a point not in A .

T or F : 15) $x \in A$ is a *limit point* of A if for each $\delta > 0$ the interval $(x - \delta, x + \delta)$ contains a point from $A \setminus \{x\}$.

T or F : 16) $x \in A$ is a *limit point* of A if for each $\delta > 0$ the interval $(x - \delta, x + \delta)$ contains a point of A .

T or F : 17) int(A) is the union of all open sets that are contained in A .

T or F : 18) $\overline{A}$ is the union of all closed sets that are contained in A .

T or F : 19) A is closed if and only if $A = \overline{A}$.

T or F : 20) A set is open if and only if it can be expressed as a countable union of disjoint open intervals.

⊛ section 4.1 – limits and continuity

T or F : 21) A is compact if and only if it is closed.

T or F : 22) The *limit of f as x approaches* x_0 *is* L if for each $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that if $x \in X$ satisfies $0 < |x - x_0| < \delta_\epsilon$ then $|f(x) - L| < \epsilon$.
Here, x_0 is a limit point of X .

T or F : 23) f is *continuous on X* if
for each $x_0 \in X$ and each $\epsilon > 0$ there is δ_{ϵ, x_0} such that
if $x_0, x \in X$ satisfies $0 < |x - x_0| < \delta_{\epsilon, x_0}$ then $|f(x) - f(x_0)| < \epsilon$.

T or F : 24) The $\lim_{x \to x_0} f(x) = L$ if and only if $f(x_n) \to L$ for each sequence $\{x_n\}$ satisfying $x_n \in X$ and $x_n \to x_0$.
Here, x_0 is a limit point of X .

T or F : 25) A function on a compact set A attains its maximum and minimum values on A .

2. Continuity

a) A function $f : X \to Y$ is *uniformly continuous on* X if for all $\epsilon > 0$

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b) Use the above $\epsilon - \delta$ definition to show that $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[1, 5]$.

⊛ Hint: $$\sqrt{x} - \sqrt{x_0} = \left[\sqrt{x} - \sqrt{x_0}\right]\left[\frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}}\right]$$

3. Let f be a continuous function from X into Y. Fix $A \subset X$. Let $B \equiv f(A) \subset Y$.

a) By definition, B is compact if and only if each open covering of B has

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b) We know that B is compact if and only if each sequence $\{b_n\}$ with $b_n \in B$ has a subsequence that converges to some number in B. Using this formulation of compactness, show that if A is compact then B is compact.

4. The LAST problem!

a) By definition, a subset U of the real line is open if for each $x \in U$ there is a $\delta_x > 0$ such that

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By definition, a subset F of the real line is closed if the complement of F is

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b) Let f be a <u>continuous</u> function from $\mathbb{R}$ into $\mathbb{R}$. Let $Z(f)$ be the zero set of f, ie. $Z(f)$ is the set of all points $p \in \mathbb{R}$ such that $f(p) = 0$, ie.

$$Z(f) \equiv \{p \in \mathbb{R} \mid f(p) = 0\} .$$

Show that $Z(f)$ is closed.