| Prof. Girardi | Math 554/703i.001 | 12.11.02 | Fall 2002 | Final Exam |
|---|---|---|---|---|

| MARK BOX | | |
|---|---|---|
| PROBLEM | POINTS | |
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| TOTAL | 40 | |
| % | 100 | |

**NAME:** ____________________

**SSN:** ______________________

**INSTRUCTIONS**:

(1) For the *proof problems*, write a NEAT FORMAL proof. You may, if you want, include your *pre-proof* but you must also include your formal proof for full credit.

(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(3) This is a closed book/notes exam covering (from *Introduction to Real Analysis*, 3rd ed., by Bartle and Sherbert): Chapters 1,2,3 .

**Problem Inspiration**:

(1,2,3) Homework to Hand-in: # 14, # 17, # 18.

(4,5,6) New Ones.

**From problems 1 – 3, do 2 of them and leave one blank. I will only grade 2 of them. You MUST indicate below which 2 you want graded and the 1 you do not want graded.**

From problems 1 – 3, I am doing problems: ____________________ .

From problems 1 – 3, I am NOT doing problem: ____________________ .

**From problems 4 – 6, do 2 of them and leave one blank. I will only grade 2 of them. You MUST indicate below which 2 you want graded and the 1 you do not want graded.**

From problems 4 – 6, I am doing problems: ____________________ .

From problems 4 – 6, I am NOT doing problem: ____________________ .

Please use your own paper or my scratch paper, indicating which problem is what and write your name on each sheet of paper. To ease in my grading, please hand-in this sheet with your test. I will post the exam on the course homepage so you can print yourself off a copy later if you want.

► **All seqeunces are understood to be sequences of real numbers.**

► **Except for problem 3, convergence of a sequence (or a series) is understood to mean convergence to a real number (and NOT $\pm\infty$).**

► I will grade harder than before - by now you should be able to write a clear proof.

**1.** Let $x_1 \geq 2$. Let $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$.

**1a.** Show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded below by 2. **1b.** Show that $\{x_n\}_{n\in\mathbb{N}}$ is decreasing.

**1c.** Conclude that $\{x_n\}_{n\in\mathbb{N}}$ converges (i.e., state a theorem that says that, given **1a** and **1b**, then you can say that $\{x_n\}_{n\in\mathbb{N}}$ converges). Find the limit of $\{x_n\}_{n\in\mathbb{N}}$.

---

**2.** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be bounded sequences.

**2a.** Show that $\liminf_{n\to\infty}(a_n + b_n) \leq (\liminf_{n\to\infty} a_n) + (\limsup_{n\to\infty} b_n)$.

**2b.** Show, by example, that in **2a** it is possible to have *strictly less than but not equal to.*

---

**3a.** Write out the definition of what it means for $\{x_n\}_{n\in\mathbb{N}}$ to **tend to** $+\infty$.

**3b.** Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences of **positive** numbers such that $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = 0$ and $\lim_{n\to\infty} x_n = +\infty$. Show that $\lim_{n\to\infty} y_n = +\infty$.

---

**4.** Let $A$ be a nonempty subset of $\mathbb{R}$ that is bounded above. Set $\alpha := \sup A$. Show that there exists (i.e., carefully construct) a sequence $\{a_n\}_{n\in\mathbb{N}}$, with $a_n \in A$ for each $n \in \mathbb{N}$, such that $\lim_{n\to\infty} a_n = \alpha$. You may use the 2 definitions below, but you may not use any fancy theorems.

**Def 1.** $\alpha := \sup A \overset{\text{i.e.}}{=} \text{l.u.b. } A$ provided:

(1) $\alpha$ is *an* upper bound of $A$ and (2) if $\beta$ is *any* upper bound of $A$, then $\alpha \leq \beta$.

**Def 2.** $\lim_{n\to\infty} a_n = \alpha \iff \forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$ s.t. if $n \geq N_\varepsilon$ then $|\alpha - a_n| < \varepsilon$.

---

**5.** Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence and $\{s_n\}_{n\in\mathbb{N}}$ be the sequence of the *averages* of the $a_n$'s, i.e.,

$$s_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \overset{\text{i.e.}}{=} \frac{1}{n}\sum_{j=1}^{n} a_j .$$

**5a.** Show that if $\{a_n\}_{n\in\mathbb{N}}$ converges, say $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} s_n = a$.

**5b.** Show, by example, that $\{s_n\}_{n\in\mathbb{N}}$ can converge although $\{a_n\}_{n\in\mathbb{N}}$ diverges.

Hint to **5a**: If $n > J \in \mathbb{N}$ then

$$|s_n - a| = \left| \left( \frac{1}{n}\sum_{j=1}^{J} a_j + \frac{1}{n}\sum_{j=J+1}^{n} a_j \right) - \left( \frac{\overbrace{a + \cdots + a}^{J \text{ times so } = Ja}}{n} + \frac{\overbrace{a + \cdots + a}^{n-J \text{ times so } = (n-J)a}}{n} \right) \right| .$$

---

**6.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and set $z_n := n\,(x_n - x_{n+1})$ for $n \in \mathbb{N}$.

**6a.** Consider the following three statements:

(1) the **series** $\sum_n x_n$ converges

(2) the **sequence** $\{nx_n\}_n$ converges

(3) the **series** $\sum_n z_n$ converges.

Show that if any 2 of these 3 statements are true, then all 3 of them are true.

**6b.** If $\sum_{n=1}^{\infty} x_n = a$ and $\lim_{n\to\infty} nx_n = b$, what is $\sum_{n=1}^{\infty} n\,(x_n - x_{n+1})$?

**6c.** Give an example of a sequence $\{x_n\}_{n\in\mathbb{N}}$ where (1) is true but (2) and (3) are false.

**6d.** Give an example of a sequence $\{x_n\}_{n\in\mathbb{N}}$ where (2) is true but (1) and (3) are false.

**6e.** Give an example of a sequence $\{x_n\}_{n\in\mathbb{N}}$ where (3) is true but (1) and (2) are false.