Prof. Girardi	Math 554/703i.001	11.25.02	Fall 2002	Exam 3

MARK BOX		
PROBLEM	POINTS	
0	10	
1	10	
2	10	
3	10	
4	10	
TOTAL	40	
%	100	

NAME: ______________________

SSN: ________________________

INSTRUCTIONS:

(1) For the *proof problems*, write a NEAT FORMAL proof. You may, if you want, include your *thinking scratch work* but you must also include your formal proof for full credit.
(2) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.
(3) This is a closed book/notes exam covering (from *Introduction to Real Analysis*, 3rd ed., by Bartle and Sherbert): lim sup, lim inf, § 3.3, 3.4, 3.5, 3.6 .

Problem Inspiration:

(1) Highly Recommended Homework § 3.3 # 1.
(2) Homework to Hand-in § 3.5 # 4.
(3&4) New Ones.

0. If the statement is true, then circle T. If the statement is false, then circle F. The scoring is 1 point for a correct answer and 0 points for an incorrect (or left blank) answer. Here, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences of real numbers.

1. T F : Bolzano and Weierstrass were cool.
2. T F : A monotone sequence of real numbers converges.
3. T F : If $\{x_n\}_{n\in\mathbb{N}}$ converges to 17, then each subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to 17.
4. T F : If $\{x_n\}_{n\in\mathbb{N}}$ is bounded, then $\lim_{n\to\infty} x_n$ exists.
5. T F : If $\{x_n\}_{n\in\mathbb{N}}$ is bounded, then $\limsup_{n\to\infty} x_n$ exists.
6. T F : $\{x_n\}_{n\in\mathbb{N}}$ converges (to a finite number) if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.
7. T F : If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded, then $\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \le \limsup_{n\to\infty}(x_n + y_n)$.
8. T F : $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
9. T F : If $x_n \le y_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = +\infty$, then $\lim_{n\to\infty} x_n = +\infty$.

0. T F : If $x_n \le y_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} y_n = +\infty$.

From problems 1 – 4, do 3 of them and leave one blank. I will only grade 3 of them. You MUST indicate below which 3 you want graded and the 1 you do not want graded.

I am doing problems: ______________________________ .

I am NOT doing problem: ______________________________ .

Please use your own paper for problems 1–4, indicating which problem is what and write your name on each sheet of paper.

▶ **All seqeunces are understood to be sequences of real numbers.**

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1. Let $x_1 := 8$ and

$$x_{n+1} \;:=\; \frac{x_n}{2} \;+\; 2$$

for $n \in \mathbb{N}$. Show that

1a. Show (hint: by induction) that $\{x_n\}_{n\in\mathbb{N}}$ is bounded below (hint: by zero will do).

1b. Show (hint: by induction) that $\{x_n\}_{n\in\mathbb{N}}$ is decreasing.

1c. Conclude that $\{x_n\}_{n\in\mathbb{N}}$ converges (i.e., state a theorem that says that, given **1a** and **1b**, then you can say that $\{x_n\}_{n\in\mathbb{N}}$ converges). Find the limit of $\{x_n\}_{n\in\mathbb{N}}$.

2a. Write out the definition of what it means for $\{x_n\}_{n\in\mathbb{N}}$ to be a **Cauchy** sequence.

2b. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be Cauchy sequences. Show that $\{x_n y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Use may use the definition in **2a** and the fact that a Cauchy sequence is bounded. You may not use the Cauchy Convergence Criterion (which says that a sequence is Cauchy $\Leftrightarrow$ it is convergent).

3a. Write out the definition of what it means for $\{x_n\}_{n\in\mathbb{N}}$ to **tend to** $+\infty$.

3b. Let $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = -17$. Show that $\lim_{n\to\infty}(x_n + y_n) = +\infty$.

4•. Recall the Bolzano-Weierstrass Theorem:
If $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence, then there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$ so that $\{a_{n_k}\}_{k\in\mathbb{N}}$ converges, say $\lim_{k\to\infty} a_{n_k} = a$.

4•. Recall the Nested Interval Property:
If $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, are nested (i.e., $I_{n+1} \subseteq I_n$) closed bounded intervals, then there exists real number $a \in \cap_{n\in\mathbb{N}} I_n$.

4. Recall that in class, we used the Nested Interval Property to prove the Bolzano-Weierstrass Theorem. Now go the other way: use the Bolzano-Weierstrass Theorem to prove the Nested Interval Property (i.e., assume that the Bolzano-Weierstrass Theorem holds and show the Nested Interval Property.) Hint: my notation above is suggestive of the proof.
