Math 550A – INFORMAL SUMMARY – Spring 2000

Ch 1: Geometry

1.1. Vector - deterermined by its magnitude (i.e. length) and direction.

Remark: a POINT $P = (a, b, c)$ determines a VECTOR $\overrightarrow{OP} = \langle a, b, c\rangle$

1.2. Products Given: $\vec{A} = \langle a_1, a_2, a_3\rangle$ and $\vec{B} = \langle b_1, b_2, b_3\rangle$

$0 \leqslant \measuredangle\vec{A}\vec{B} \stackrel{\text{def}}{=}$ angle between $\vec{A}$ & $\vec{B} \stackrel{\text{def}}{=} \theta_{AB} \leqslant \pi$

$\vec{n}_{AB} \stackrel{\text{def}}{=}$ the right-hand-rule unit vector $\perp$ to $\vec{A}$ and $\vec{B}$

direction angles $0 \leqslant \alpha, \beta, \gamma \leqslant \pi$ for $\vec{A}$.

Then:

$$\text{length of } \vec{A} = \left\|\vec{A}\right\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\vec{A}\bullet\vec{A}}$$

$$\text{dot product} = \vec{A}\bullet\vec{B} = \left\|\vec{A}\right\| \left\|\vec{B}\right\| \cos\theta_{AB} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\text{cross product} = \vec{A}\times\vec{B} = \left[\left\|\vec{A}\right\| \left\|\vec{B}\right\| \sin\theta_{AB}\right] \vec{n}_{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\text{triple product} = \left[\vec{A}, \vec{B}, \vec{C}\right] = \vec{A}\times\vec{B}\bullet\vec{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{angle btw.} = \measuredangle\vec{A}\vec{B} = \cos^{-1}\left(\frac{\vec{A}\bullet\vec{B}}{\left\|\vec{A}\right\| \left\|\vec{B}\right\|}\right)$$

$$\alpha = \measuredangle\vec{A}\,\vec{i} = \cos^{-1}\left(a_1 \,/\, \left\|\vec{A}\right\|\right)$$

$$\beta = \measuredangle\vec{A}\,\vec{j} = \cos^{-1}\left(a_2 \,/\, \left\|\vec{A}\right\|\right)$$

$$\gamma = \measuredangle\vec{A}\,\vec{k} = \cos^{-1}\left(a_3 \,/\, \left\|\vec{A}\right\|\right) \quad \text{so } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

1.3. 1.2 in action $\vec{A} \neq \vec{0}$ and $\vec{B} \neq \vec{0}$

$\vec{A}$ is a UNIT vector $\Leftrightarrow$ $\left\|\vec{A}\right\| = 1$

the NORMALIZED vector of $\vec{A}$ is $\frac{\vec{A}}{\|\vec{A}\|}$

$\vec{A}$ & $\vec{B}$ are ORTHONORMAL $\leftrightarrow$ $\vec{A} \perp \vec{B}$ and $\left\|\vec{A}\right\| = 1 = \left\|\vec{B}\right\|$

$\vec{A} \perp \vec{B}$ $\leftrightarrow$ $\vec{A}$ & $\vec{B}$ are PERPENDICULAR $\leftrightarrow$ $\vec{A}\bullet\vec{B} = 0$ $\leftrightarrow$ $\vec{A}$ & $\vec{B}$ are ORTHOGONAL

$\vec{A}\|\vec{B}$ $\leftrightarrow$ $\vec{A}$ & $\vec{B}$ are PARELLEL $\leftrightarrow$ $\vec{A}\times\vec{B} = \vec{0}$ $\leftrightarrow$ $\vec{A} = k\vec{B}$ for some $k \in \mathbb{R}$

the area of the triangle determined by $\vec{A}$ and $\vec{B}$ is $Area = \frac{1}{2}\left\|\vec{A}\times\vec{B}\right\|$

the area of the parallelogram determined by $\vec{A}$ and $\vec{B}$ is $Area = \left\|\vec{A}\times\vec{B}\right\|$

the volume of the parallelepiped determined by $\vec{A}, \vec{B}, \vec{C}$ is $Vol = \left|\,\left[\vec{A}, \vec{B}, \vec{C}\right]\,\right|$

1.4. Algebraic Properties

$$\left(s\vec{A}\right) \bullet \vec{B} = s\left(\vec{A} \bullet \vec{B}\right) = \vec{A} \bullet \left(s\vec{B}\right) \quad \leftrightsquigarrow \quad \left(s\vec{A}\right) \times \vec{B} = s\left(\vec{A} \times \vec{B}\right) = \vec{A} \times \left(s\vec{B}\right)$$

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A} \quad \leftrightsquigarrow \quad \vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$$

$$\left(\vec{A} + \vec{B}\right) \bullet \vec{C} = \vec{A} \bullet \vec{C} + \vec{B} \bullet \vec{C} \quad \leftrightsquigarrow \quad \left(\vec{A} + \vec{B}\right) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$$

$$\vec{A} \bullet \left(\vec{B} + \vec{C}\right) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C} \quad \leftrightsquigarrow \quad \vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

1.5. Components

The parallel-perpendicular decomposition of $\vec{B}$ is $\vec{B} = \vec{B}_{\|} + \vec{B}_{\perp}$ where:

$$\vec{B}_{\|} = \frac{\vec{A} \bullet \vec{B}}{\vec{A} \bullet \vec{A}} \vec{A} = \left(\vec{B} \bullet \frac{\vec{A}}{\|\vec{A}\|}\right) \frac{\vec{A}}{\|\vec{A}\|} \qquad \text{is } \| \text{ to } \vec{A}$$

$$\vec{B}_{\perp} = \frac{(\vec{A} \times \vec{B}) \times \vec{A}}{\vec{A} \bullet \vec{A}} = \vec{B} - \vec{B}_{\|} \qquad \text{is } \perp \text{ to } \vec{A} \, .$$

$\vec{B}_{\|}$ is called the orthogonal projection of $\vec{B}$ onto $\vec{A}$.

The (signed) component of $\vec{B}$ along $\vec{A}$ is:

$$\text{comp}_{\vec{A}}\vec{B} \overset{\text{def}}{=} \begin{bmatrix} + \left\|\vec{B}_{\|}\right\| & \text{if } 0 \leqslant \theta_{AB} \leqslant \pi/2 \\ - \left\|\vec{B}_{\|}\right\| & \text{if } \pi/2 \leqslant \theta_{AB} \leqslant \pi \end{bmatrix} \overset{1.2}{=} \left\|\vec{B}\right\| \cos\theta_{AB} \overset{1.2}{=} \vec{B} \bullet \frac{\vec{A}}{\left\|\vec{A}\right\|} \, .$$

1.6. Equations and parameterizations

▷ The line $\mathcal{L}$ through $R_0 = (x_0, \ y_0, \ z_0)$ and parallel to $\vec{v}_0 = \langle a, \ b, \ c\rangle$:

$$\vec{R}(t) = \vec{R}_0 + t\vec{v}_0 \iff \begin{array}{l} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{array} \iff \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad \text{if } abc \neq 0$$

▷ The plane $\mathcal{P}$ through $R_0 = (x_0, \ y_0, \ z_0)$ and with normal $\vec{n} = \langle a, \ b, \ c\rangle$:

$$\left(\langle x, y, z\rangle - \vec{R}_0\right) \bullet \vec{n} = 0 \iff a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\iff ax + by + cz = d \text{ where } d = ax_0 + by_0 + cz_0 \, .$$

▷ For a plane $\mathcal{P}$ given by $ax + by + cz = d$:

$$d\left(\mathcal{P}, (x_1, y_1, z_1)\right) = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

▷ For radius ρ, center R_0, and a right-handed system of orthonormal vector $\vec{e}_1, \vec{e}_2, \vec{e}_3$:

$$\vec{R}(t) = \vec{R}_0 + \cos t \, \rho \, \vec{e}_1 + \sin t \, \rho \, \vec{e}_2 + at \, \vec{e}_3$$

- if $a = 0$: is a circle in the plane determined by $\vec{e}_1$ and $\vec{e}_2$
- if $a > 0$: is a right–handed helix with pitch $2\pi \, |a|$ and axis parallel to $\vec{e}_3$
- if $a < 0$: is a left–handed helix with pitch $2\pi \, |a|$ and axis parallel to $\vec{e}_3$.

1.7. <u>Force</u>: For a constant force $\vec{F}$ acting thru a displacement $\vec{D}$

$\triangleright$ The work W done by $\vec{F}$ is $\qquad W = \vec{D} \bullet \vec{F}$

$\triangleright$ The torque τ due to $\vec{F}$ at the point D is $\qquad \tau = \vec{D} \times \vec{F}$.

1.8. <u>Cylindrical Coordinates</u> (r, θ, z) of (x, y, z) :

$$r \geqslant 0 \qquad 0 \leqslant \theta < 2\pi \qquad -\infty < z < \infty$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \begin{cases} \tan^{-1}(y/x) & \text{if } (x,y) \in 1^{\text{st}} \text{ quad} \\ \pi + \tan^{-1}(y/x) & \text{if } (x,y) \in 2/3^{\text{nd}} \text{ quad} \\ 2\pi + \tan^{-1}(y/x) & \text{if } (x,y) \in 4^{\text{th}} \text{ quad} \end{cases} \qquad z = z$$

1.9. <u>Spherical Coordinates</u> (ρ, θ, ϕ) of (x, y, z) : note $r = \rho \sin\phi$

$$\rho \geqslant 0 \qquad 0 \leqslant \theta < 2\pi \qquad 0 \leqslant \phi \leqslant \pi$$

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \text{as above} \qquad \phi = \cos^{-1}\frac{z}{\rho}$$

Figure 1.4.2

Representing a point (x, y, z) in Cartesian coordinates in its cylindrical coordinates (r, θ, z).

Figure 1.4.5

Representing a point (x, y, z) in Cartesian coordinates in its spherical coordinates (ρ, θ, ϕ).

Ch 2: Differentiation & §4.1–4.2 Motion

2.1. Defs.

a. the neighborhood of $p \in \mathbb{R}^n$ of radius r is $N_r(p) = \{q \in \mathbb{R}^n : \|p - q\| < r\}$

b. open set - pg 94

2.2. Set-up

Given: an open subset $U^n \subset \mathbb{R}^n$

a function $\vec{f}: U^n \to \mathbb{R}^m$.

So elts. of U^n look like $\vec{x} = \langle x_1, \ldots, x_n \rangle$ and $\vec{x}_0 = \langle x_{01}, \ldots, x_{0n} \rangle$

and $\vec{f}$ has the form

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}_{m \times 1}$$

where $f_i: U^n \to \mathbb{R}^1$.

► Limits (pg 98) and Continuity (pg 104)

$\lim_{\vec{x} \to \vec{x}_0} \vec{f}(\vec{x}) = \vec{b} \overset{\text{def}}{\iff} \vec{f}(\vec{x}) \approx \vec{b}$ provided $\vec{x} \approx \vec{x}_0$

$\iff \lim_{\vec{x} \to \vec{x}_0} \langle f_1(\vec{x}), \ldots, f_m(\vec{x}) \rangle = \langle b_1, \ldots, b_m \rangle$

$\iff \lim_{\vec{x} \to \vec{x}_0} f_i(\vec{x}) = b_i$ for each i .

$\vec{f}$ is continuous at $\vec{x}_0 \overset{\text{def}}{\iff} \lim_{\vec{x} \to \vec{x}_0} \vec{f}(\vec{x}) = \vec{f}(\vec{x}_0)$

$\iff \lim_{\vec{x} \to \vec{x}_0} \langle f_1(\vec{x}), \ldots, f_m(\vec{x}) \rangle = \langle f_1(\vec{x}_0), \ldots, f_m(\vec{x}_0) \rangle$

$\iff \lim_{\vec{x} \to \vec{x}_0} f_i(\vec{x}) = f_i(\vec{x}_0)$ for each i

$\iff$ each f_i is continuous at $\vec{x}_0$.

► Partial Derivatives of the f_i's and the class C^k

Def: An *iterated partial derivative* (i.p.d.) of f_i of order k is of the form:

$$\frac{\partial^k f_i}{\partial x_{i_k} \cdots \partial x_{i_2} \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \left(\cdots \left(\frac{\partial}{\partial x_{i_2}} \left(\frac{\partial f_i}{\partial x_{i_1}} \right) \right) \right) .$$

Def: $\vec{f}$ is (in the class) C^k at $\vec{x}_0 \in U^n$

$\Leftrightarrow$ each of the k^{th}-ordered i.p.d. of each f_i exists and is continuous in a neighborhood of $\vec{x}_0$.

Def: $\vec{f}$ is (in the class) C^k on U^n

$\Leftrightarrow$ $\vec{f}$ is C^k at each $\vec{x} \in U^n$

$\Leftrightarrow$ each of the k^{th}-ordered i.p.d. of each f_i exists and is continuous at each $\vec{x} \in U^n$.

Euler's Theorem **3.1**: If $f_i \in C^2$ at $\vec{x}_0 \in U^n$, then $\frac{\partial^2 f_i(\vec{x}_0)}{\partial x_{i_2} \partial x_{i_1}} = \frac{\partial^2 f_i(\vec{x}_0)}{\partial x_{i_1} \partial x_{i_2}}$

▶ <u>Derivative of f</u>

Form: the matrix of partial derivatives of $\vec{f}$,

which is also called the <u>the derivative of $\vec{f}$ (at $\vec{x}_0$)</u> :

$$D_M \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m\times n} \quad \leftrightsquigarrow \quad D_M \vec{f}(\vec{x}_0) = \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial x_1} & \cdots & \frac{\partial f_1(\vec{x}_0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\vec{x}_0)}{\partial x_1} & \cdots & \frac{\partial f_m(\vec{x}_0)}{\partial x_n} \end{bmatrix}_{m\times n} .$$

and consider

$$D_M \vec{f}(\vec{x}_0) \otimes (\vec{x} - \vec{x}_0) \overset{\text{def}}{=} \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial x_1} & \cdots & \frac{\partial f_1(\vec{x}_0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\vec{x}_0)}{\partial x_1} & \cdots & \frac{\partial f_m(\vec{x}_0)}{\partial x_n} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 - x_{01} \\ \vdots \\ x_n - x_{0n} \end{bmatrix}_{n\times 1}$$

$$= \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial x_1}(x_1 - x_{01}) + \ldots + \frac{\partial f_1(\vec{x}_0)}{\partial x_n}(x_n - x_{0n}) \\ \vdots \\ \frac{\partial f_m(\vec{x}_0)}{\partial x_1}(x_1 - x_{01}) + \ldots + \frac{\partial f_m(\vec{x}_0)}{\partial x_n}(x_n - x_{0n}) \end{bmatrix}_{m\times 1} .$$

Then <u>$\vec{f}$ is differentiable at $\vec{x}_0$</u> if the partials $\frac{\partial f_i}{\partial x_j}$ of $\vec{f}$ exist at $\vec{x}_0$ and if

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\left\| \vec{f}(\vec{x}) - \left[\vec{f}(\vec{x}_0) + D_M \vec{f}(\vec{x}_0) \otimes (\vec{x} - \vec{x}_0) \right] \right\|}{\|\vec{x} - \vec{x}_0\|} = 0 . \qquad (2.2')$$

When this is the case, if $\vec{x} \approx \vec{x}_0$, then:

$$\vec{f}(\vec{x}) \approx \vec{f}(\vec{x}_0) + D_M \vec{f}(\vec{x}_0) \otimes (\vec{x} - \vec{x}_0) , \qquad (2.2'')$$

i.e.,

$$\begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_0) \\ \vdots \\ f_m(\vec{x}_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(\vec{x}_0)}{\partial x_1}(x_1 - x_{01}) + \ldots + \frac{\partial f_1(\vec{x}_0)}{\partial x_n}(x_n - x_{0n}) \\ \vdots \\ \frac{\partial f_m(\vec{x}_0)}{\partial x_1}(x_1 - x_{01}) + \ldots + \frac{\partial f_m(\vec{x}_0)}{\partial x_n}(x_n - x_{0n}) \end{bmatrix} .$$

⇝. <u>§ 2.3 Theorem 9</u>: If $\vec{f} \in C^1$ at $\vec{x}_0 \in U^n$, then $\vec{f}$ is differentiable at $\vec{x}_0$.
If $\vec{f} \in C^1$ at each $\vec{x}_0 \in U^n$, then $\vec{f}$ is differentiable on U^n.

2.3. Special Cases of (2.2″): $\underline{f\colon \mathbb{R} \to \mathbb{R}}$ so: $y = f(x)$ and $D_{M} f = \left[\, \frac{df}{dx} \,\right]$.

If $x \approx x_0$ then:

$$\begin{aligned} f(x) &\approx f(x_0) + \left[\, \tfrac{df(x_0)}{dx} \,\right] \otimes [\, x - x_0 \,] \\ &\approx f(x_0) + \frac{df(x_0)}{dx}(x - x_0) \,. \end{aligned}$$

Special Cases of (2.2″): $\underline{f\colon \mathbb{R}^2 \to \mathbb{R}}$ so: $z = f(x, y)$ and $D_{M} f = \left[\, \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \,\right]$.

If $(x, y) \approx (x_0, y_0)$ then:

$$\begin{aligned} f(x,y) &\approx f(x_0, y_0) + \left[\, \tfrac{\partial f(x_0,y_0)}{\partial x} \quad \tfrac{\partial f(x_0,y_0)}{\partial y} \,\right] \otimes \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \\ &\approx f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}\,(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}\,(y - y_0) \,. \end{aligned}$$

2.4. Differentiation Rules

Given: an open subset $U^n \subset \mathbb{R}^n$ and $\vec{x}_0 \in U^n$

functions $\vec{f}\colon U^n \to \mathbb{R}^m$ and $\vec{g}\colon U^n \to \mathbb{R}^m$ that are each differentiable at $\vec{x}_0$

a constant $c \in \mathbb{R}$.

Then:

$$\begin{aligned} D_{M}\left[c\, \vec{f}\right](\vec{x}_0) &= c\, D_{M}\left[\vec{f}(\vec{x}_0)\right] \\ D_{M}\left[\vec{f} + \vec{g}\right](\vec{x}_0) &= D_{M}\left[\vec{f}(\vec{x}_0)\right] + D_{M}\left[\vec{g}(\vec{x}_0)\right] \\ D_{M}\left[\vec{f}\vec{g}\right](\vec{x}_0) &\overset{m=1}{=\!=} D_{M}\left[\vec{f}(\vec{x}_0)\right]\, \vec{g}(\vec{x}_0) + \vec{f}(\vec{x}_0)\, D_{M}\left[\vec{g}(\vec{x}_0)\right] \\ D_{M}\left[\frac{\vec{f}}{\vec{g}}\right](\vec{x}_0) &\underset{\vec{g}(\vec{x}_0)\neq 0}{\overset{m=1}{=\!=\!=}} \frac{D_{M}\left[\vec{f}(\vec{x}_0)\right]\, \vec{g}(\vec{x}_0) - \vec{f}(\vec{x}_0)\, D_{M}\left[\vec{g}(\vec{x}_0)\right]}{\left[\vec{g}(\vec{x}_0)\right]^2} \,. \end{aligned}$$

Given: open subsets $U^n \subset \mathbb{R}^n$ and $V^m \subset \mathbb{R}^m$ and $\vec{x}_0 \in U^n$

functions $\vec{g}\colon U^n \to \mathbb{R}^m$ and $\vec{f}\colon V^m \to \mathbb{R}^p$

$\vec{g}(U^n) \subset V^m$ and so it makes sense to have $\vec{f} \circ \vec{g}\colon U^n \to \mathbb{R}^p$.

$\vec{g}$ is differentiable at $\vec{x}_0$ and $\vec{f}$ is differentiable at $\vec{g}(\vec{x}_0)$.

Then the Chain Rule says that :

$$D_{M}\left(\vec{f} \circ \vec{g}\right)(\vec{x}_0) = D_{M}\vec{f}\left(\vec{g}\left(\vec{x}_0\right)\right) \otimes D_{M}\vec{g}(\vec{x}_0) \,.$$

2.5. Vector-Valued Functions

Given: an subset $\mathbb{D}^1 \subset \mathbb{R}^1$ and $t_0 \in \mathbb{D}^1$

a function $\vec{F} \colon \mathbb{D}^1 \to \mathbb{R}^m$

so $\vec{F}$ has the form $\boxed{\vec{F}(t) \;=\; \langle f_1(t), \ldots, f_m(t) \rangle}$ where $f_i \colon \mathbb{D}^1 \to \mathbb{R}$.

$\vec{F}$ is called a VECTOR-VALUED FUNCTION .

One thinks of $\vec{F}$ as tracing out (ie. parameterizing) a (1-D) space curve $\mathcal{C}$ in $\mathbb{R}^m$.

When $\mathbb{D}^1$ is an open set:

$$D_{\scriptscriptstyle M} \vec{F} \;=\; \begin{bmatrix} \frac{df_1}{dt} \\ \vdots \\ \frac{df_m}{dt} \end{bmatrix}$$

and (2.2′) becomes:

$$\vec{F} \text{ is differentiable at } t_0 \qquad \Longleftrightarrow \qquad \lim_{t \to t_0} \frac{\vec{F}(t) \,-\, \vec{F}(t_0)}{t - t_0} \quad \text{exists}$$

and in this case,

$$\frac{d\vec{F}}{dt}\Big|_{t_0} \;=\; \vec{F}'(t_0) \;=\; \lim_{t \to t_0} \frac{\vec{F}(t) \,-\, \vec{F}(t_0)}{t - t_0} \;=\; \langle f_1'(t_0), \ldots, f_m'(t_0) \rangle \; .$$

2.6. Paths/Curves

▷ A PATH is a vector-valued function $\vec{r} \colon\ [a,b] \to \mathbb{R}^m$. Usually, $m = 2 \text{or} 3$.

▷ The CURVE or ARC $\mathcal{C}$ that is PARAMETERIZED by $\vec{r}$ is $\mathcal{C} \;=\; \{\vec{r}(t) \colon a \leqslant t \leqslant b\}$.

▷ The unit tangent vector to $\mathcal{C}$ at t_0 is $\vec{T} = \frac{\vec{r}\,'(t_0)}{\|\vec{r}\,'(t_0)\|}$.

▷ If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then the arc length of $\mathcal{C}$ is: $s(a,b) \;=\; \int_a^b ds \;=\; \int_a^b \|\vec{r}\,'(t)\|\, dt$

w/ $d\vec{s} \;=\; \vec{r}\,'(t)\, dt \;=\; \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle\, dt$ & $ds \;=\; \|d\vec{s}\| \;=\; \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$

▷ If: $I \subset [a,b]$

then: $\vec{r}$ is 1-to-1 on I $\Leftrightarrow$ the curve $\{\vec{r}(t) \colon t \in I\}$ does not intersect itself.

▷ $\mathcal{C}$ is a SIMPLE CURVE if it has a parameterization $\vec{r}$ where:

(1) $\vec{r}$ is piecewise C^1 on $[a,b]$ (2) $\vec{r}$ is 1-to-1 on $[a,b]$.

▷ $\mathcal{C}$ is a CLOSED CURVE if $\vec{r}(a) = \vec{r}(b)$.

▷ $\mathcal{C}$ is a SIMPLE CLOSED CURVE if it has a parameterization $\vec{r}$ where:

(1) $\vec{r}$ is piecewise C^1 on $[a,b]$ (2) $\vec{r}$ is 1-to-1 on $[a,b)$

(3) $\vec{r}(a) = \vec{r}(b)$.

▷ $\mathcal{C}$ is a SMOOTH if it has a parameterization $\vec{r}$ where:

(1) $\vec{r}$ is C^1 on $[a,b]$ (2) $\vec{r}$ is 1-to-1 on $[a,b)$

(3) $\vec{r}\,' \neq \vec{0}$ on $[a,b]$.

2.7. Differentiation Rules for vector-valued functions

Given: an open subset $\mathbb{D}^1 \subset \mathbb{R}^1$

differentiable functions $\vec{F}, \vec{G} \colon \mathbb{D}^1 \to \mathbb{R}^m$ and $p \colon \mathbb{D}^1 \to \mathbb{R}$

Then:
$$\tfrac{d}{dt}\Big[p(t)\,\vec{F}(t)\Big] = p'(t)\vec{F}(t) + p(t)\vec{F}'(t)$$
$$\tfrac{d}{dt}\Big[\vec{F}(t) \bullet \vec{G}(t)\Big] = \vec{F}'(t) \bullet \vec{G}(t) + \vec{F}(t) \bullet \vec{G}'(t)$$
$$\tfrac{d}{dt}\Big[\vec{F}(t) \times \vec{G}(t)\Big] = \vec{F}'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t) .$$

Chain Rule: $$\tfrac{d}{dt}\Big[\vec{F}(p(t))\Big] = p'(t)\vec{F}'(p(t)) .$$

2.8. Motion For a particle moving through space on a "nice" curve $\mathcal{C}$ with :

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ position vector

$\vec{v}(t) = \dfrac{d\vec{r}(t)}{dt}$ velocity vector

$\vec{a}(t) = \dfrac{d\vec{v}(t)}{dt}$ acceleration vector

$v(t) = \|\vec{v}(t)\| \neq \dfrac{d\,\|\vec{r}(t)\|}{dt}$ speed function

$\vec{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|}$ unit tangent vector to $\mathcal{C}$ at the pt. $\vec{r}(t)$

$\vec{N}(t) = \dfrac{d\vec{T}(t)/dt}{\left\|d\vec{T}(t)/dt\right\|}$ unit principle normal vector to $\mathcal{C}$ at the pt. $\vec{r}(t)$

$\vec{T}(t) \perp \vec{N}(t)$ with $\vec{T}(t) \parallel \vec{v}(t)$ & $\vec{N}(t) \perp \vec{v}(t)$

$\vec{a}(t) = a_T(t)\,\vec{T}(t) + a_N(t)\,\vec{N}(t)$ with $a_T(t) = \dfrac{dv(t)}{dt}$ & $a_N(t) = v(t)\left\|\dfrac{d\vec{T}(t)}{dt}\right\|$

$\|\vec{a}(t)\|^2 = |a_T(t)|^2 + |a_N(t)|^2$ & $a_N \geqslant 0$

$\vec{F} = m\vec{a}$ if a force $\vec{F}$ is acting on the particle of mass m

2.9. Scalar Fields

Given: a subset $\mathbb{D}^n \subset \mathbb{R}^n$ and a function $f \colon \mathbb{D}^n \to \mathbb{R}$

so f has the form $f(\vec{x}) = w$, or equiv. $f(x_1, \ldots, x_n) = w$

Then: f is called a SCALAR FIELD

the GRAPH of $f = \big\{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \in \mathbb{R}^{n+1} \colon (x_1, \ldots, x_n,) \in \mathbb{D}^n\big\}$

the LEVEL SET of value $c \in \mathbb{R}$ for f is $\{\vec{x} \in \mathbb{D}^n \colon f(\vec{x}) = c\} \subset \mathbb{R}^n$

▷ $[n = 2] \Rightarrow$ [level set = LEVEL CURVE]

▷ $[n = 3] \Rightarrow$ [level set = LEVEL SURFACE]

$D_M f = \left[\frac{\partial f}{\partial x_1} \quad \ldots \quad \frac{\partial f}{\partial x_n}\right]$

the GRADIENT of $f = \overrightarrow{\nabla} f = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$.

2.10. Curves on Surfaces

★ Given: an open set $\mathbb{D}^n \subset \mathbb{R}^n$ and a scalar field $f : \mathbb{D}^n \to \mathbb{R}$

$f(x_1, \dots, x_n) \ = \ w$

an interval $I \subset \mathbb{R}$ and a path $\vec{r}\colon I \to \mathbb{R}^n$

$\vec{r}(t) = \langle r_1(t), \dots, r_n(t) \rangle$

$\vec{r}(I) \subset \mathbb{D}^n$ so $F \stackrel{\text{def}}{=\!=} f \circ \vec{r}\colon I \to \mathbb{R}$

$F(t) = f(r_1(t), \dots, r_n(t))$

When $n = 2$, think of F as a curve on the surface of f.

If f and $\vec{r}$ are *nice*, ie. differentiable, then

$$D_{M} f \ = \ \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \qquad \text{and} \qquad D_{M} \vec{r} \ = \ \begin{bmatrix} \frac{dr_1}{dt} \\ \vdots \\ \frac{dr_n}{dt} \end{bmatrix}$$

and the Chain Rule (2.4) gives:

$$\frac{dF}{dt}\Big|_{t_0} \ = \ D_{M} f\,(\vec{r}\,(t_0)) \ \otimes \ D_{M} \vec{r}\,(t_0) \ = \ \vec{\nabla} f\big|_{\vec{r}(t_0)} \bullet \ \vec{r}\,'(t_0)\ .$$

► In ★, consider the level set $\mathcal{S}$ given by $f(\vec{x}) = k$:

If: $\vec{r}(I) \subset \mathcal{S}$ so $(f \circ \vec{r}\,)\,(t) = k$ for each $t \in I$

$\vec{x}_0 \in \mathcal{S}$ so $f(\vec{x}_0) = k$

$t_0 \in I$ with $\vec{r}(t_0) \ = \ \vec{x}_0$

Then: $\vec{\nabla} f\big|_{\vec{x}_0} \bullet \vec{r}\,'(t_0) \ = \ \frac{d}{dt}\,(f \circ \vec{r})\,\big|_{t_0} \ = \ 0$

so $\vec{\nabla} f\big|_{\vec{x}_0} \ \perp \ $ (the tangent vector to $\vec{r}(t)$) $\big|_{t_0}$

so $\vec{\nabla} f\big|_{\vec{x}_0} \ \perp \ \mathcal{S}\big|_{\vec{x}_0}$.

► In ★, consider the case where $\vec{r}$ is a line segment through $\vec{x}_0$ in the direction of $\vec{u}$:

Given: $\vec{x}_0 \in \mathbb{D}^n$ and $\vec{u} \in \mathbb{R}^n$ and so $\vec{r}(t) = \vec{x}_0 + t\vec{u}$.

The <u>DIRECTIONAL DERIVATIVE</u> of f at $\vec{x}_0$ along $\vec{u}$ is :

$$D_{\vec{u}} f(\vec{x}_0) \ \stackrel{\text{def}}{=\!=} \ \frac{d(f \circ \vec{r})}{dt}\Big|_{\substack{\vec{x}=\vec{x}_0 \\ t=0}} \ = \ \lim_{h \to 0} \frac{f\,(\vec{x}_0 + h\vec{u}) - f\,(\vec{x}_0)}{h}$$

$$= \ \vec{u} \bullet \ \vec{\nabla} f\big|_{\vec{x}_0} \qquad = \ \|\vec{u}\| \ \Big\| \vec{\nabla} f\big|_{\vec{x}_0} \Big\| \ \cos \measuredangle \vec{u}\,\vec{\nabla} f\big|_{\vec{x}_0}$$

When $\|\vec{u}\| = 1$, since $D_{\vec{u}} f\big|_{\vec{x}_0}$ is just the rate of change of f at $\vec{x}_0$ in the direction of $\vec{u}$,

$D_{\vec{u}} f\big|_{\vec{x}_0}$ is called the directional derivative of f at $\vec{x}_0$ in the direction of $\vec{u}$.

$\pm \ \vec{\nabla} f$ points in the direction of the maximum rate of $\left[\begin{smallmatrix}\text{increase}\\ \text{decrease}\end{smallmatrix}\right]$ of f .

$\pm \Big\| \vec{\nabla} f \Big\|$ equals the maximum rate of $\left[\begin{smallmatrix}\text{increase}\\ \text{decrease}\end{smallmatrix}\right]$ of f per unit distance .

2.11. Tangent Planes

Here: open sets $\mathbb{D}^n \subset \mathbb{R}^n$

► Given: a scalar field $f\colon \mathbb{D}^3 \to \mathbb{R}$, so f has the form $w = f(x,y,z)$

$\vec{x}_0 = \langle x_0, y_0, z_0 \rangle \in \mathbb{D}^3$ with $f(\vec{x}_0) = k$

the level surface $\mathcal{S}$ given by $f(\vec{x}) = k$, so $\vec{x}_0 \in \mathcal{S}$

$\vec{\nabla} f\big|_{\vec{x}_0} \neq 0$.

Then: $\vec{\nabla} f\big|_{\vec{x}_0} \perp \mathcal{S}\big|_{\vec{x}_0}$

the eq. of the tangent plane to $\mathcal{S}$ at the point (x_0, y_0, z_0) is :

$$\vec{\nabla} f\big|_{\vec{x}_0} \bullet \langle x - x_0, y - y_0, z - z_0 \rangle = 0 .$$

► Given: a scalar field $g\colon \mathbb{D}^2 \to \mathbb{R}$, so g has the form $z = g(x,y)$

Let: $f(x,y,z) = g(x,y) - z$

So: [the level surface $f(x,y,z) = 0$] $=$ [the graph (ie. surface) of $z = g(x,y)$] $\stackrel{\text{def}}{=} \mathcal{S}$

So: $\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle\big|_{(x_0,y_0)} \perp \mathcal{S}\big|_{(x_0,y_0,g(x_0,y_0))}$

the eq. of the tangent plane to $\mathcal{S}$ at the point $(x_0, y_0, g(x_0,y_0))$ is both :

$$\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle\big|_{(x_0,y_0)} \bullet \langle x - x_0, y - y_0, z - g(x_0,y_0) \rangle = 0$$

$$z = g(x_0,y_0) + \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)} (x - x_0) + \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)} (y - y_0) .$$

► Given: a scalar field $h\colon \mathbb{D}^1 \to \mathbb{R}$, so h has the form $y = h(x)$

Let: $f(x,y,z) = h(x) - y$

So: [the level surface $f(x,y,z) = 0$ intersected with the plane $z = 0$]

$=$ [the graph of $y = h(x)$] $\stackrel{\text{def}}{=} \mathcal{G}$

So: $\langle \frac{dh}{dx}, -1, 0 \rangle\big|_{(x_0)} \perp \mathcal{G}\big|_{(x_0,h(x_0),0)}$

the equation of the tangent line to $\mathcal{G}$ at the point $(x_0, h(x_0), 0)$ is both :

$$\langle \frac{dh}{dx}, -1, 0 \rangle\big|_{(x_0)} \bullet \langle x - x_0, y - h(x_0), 0 \rangle = 0$$

$$y = h(x_0) + \frac{dh}{dx}\Big|_{(x_0)} (x - x_0) \quad \text{and} \quad z = 0 .$$

§4.3–4.4 Vector Fields and Div and Curl

4.1. Vector Fields

Given: a subset $\mathbb{D}^n \subset \mathbb{R}^n$

a function $\vec{F} \colon \mathbb{D}^n \to \mathbb{R}^n$

so $\vec{F}$ has the form $\vec{F}(x_1, \ldots, x_n) = \langle F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n) \rangle$

where $F_i \colon \mathbb{D}^n \to \mathbb{R}$.

Then: $\vec{F}$ is called a vector field

visualize $\vec{F}$ as attaching the vector $\vec{F}(\vec{x})$ to the point x

F_i are called the component scalar fields

Given: a path $\vec{c} \colon \mathbb{D}^1 \to \mathbb{R}^n$

so $\vec{c}$ has the form $\vec{c}(t) = \langle x_1(t), \ldots, x_n(t) \rangle$ where $x_i \colon \mathbb{D}^1 \to \mathbb{R}$.

If: $\vec{c}\,'(t) = \vec{F}\left(\vec{c}\,(t)\right)$

Then: $\vec{c}$ is called a flow line of $\vec{F}$

$\vec{c}$ is the sol'n to the system of differential eqs: $x_i'(t) = F_i\left(x_1\left(t\right), \ldots, x_n\left(t\right)\right)$

$\vec{F}(\vec{c}(t_0))$ is tangent to the curve $\vec{c}$ at the point $\vec{c}(t_0)$

$n = 3$: if $\vec{F}$ is the velocity field of a fluid, then $\vec{c}$ is the path followed by a particle in the fluid .

4.2. Divergence & Curl

Defs: the vector differential operator $\quad \vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

the Laplacian operator $\quad \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \vec{\nabla} \bullet \vec{\nabla}$

Let: $\begin{bmatrix} \mathcal{S} \\ \mathcal{V} \end{bmatrix}$ be the class of all the (nice) $\begin{bmatrix} \text{scalar} \\ \text{vector} \end{bmatrix}$ fields from $\mathbb{D}^3 \subset \mathbb{R}^3$ to $\begin{bmatrix} \mathbb{R} \\ \mathbb{R}^3 \end{bmatrix}$.

Defs: for $f \in \mathcal{S}$ and $\vec{F} \in \mathcal{V}$

$$\overrightarrow{\text{grad}}\, f = \vec{\nabla}\,(f) \quad ; \quad \text{div}\ \vec{F} = \vec{\nabla} \bullet \vec{F} \quad ; \quad \overrightarrow{\text{curl}}\, \vec{F} = \vec{\nabla} \times \vec{F} .$$

So:

$$\mathcal{S} \xrightarrow{\overrightarrow{\text{grad}}} \mathcal{V} \xrightarrow{\overrightarrow{\text{curl}}} \mathcal{V} \xrightarrow{\text{div}} \mathcal{S}$$

$$\mathcal{S} \xrightarrow{\vec{\nabla}^2} \mathcal{S} \quad \text{and} \quad \mathcal{V} \xrightarrow{\vec{\nabla}^2} \mathcal{V}$$

Given: a function $\vec{F} \colon \mathbb{D}^2 \to \mathbb{R}^2$

so $\vec{F}$ has the form $\vec{F} = \langle F_1, F_2 \rangle$

Then: scalar curl of $\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

4.3. Basic Identities

1. $\vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g$
2. $\vec{\nabla}(cf) = c\vec{\nabla} f$ where $c \in R$
3. $\vec{\nabla}(fg) = f\vec{\nabla} g + g\vec{\nabla} f$
4. $\vec{\nabla}\left(\frac{f}{g}\right) = \frac{g\vec{\nabla} f - f\vec{\nabla} g}{g^2}$ at points $\vec{x}$ where $g(\vec{x}) \neq 0$
5. div $(\vec{F}+\vec{G}) =$ div $\vec{F}$ + div $\vec{G}$
6. $\overrightarrow{\text{curl}}(\vec{F}+\vec{G}) = \overrightarrow{\text{curl}}\vec{F} + \overrightarrow{\text{curl}}\vec{G}$
7. div $(f\vec{F}) = f$ div $\vec{F} + \vec{F} \bullet \vec{\nabla} f$
8. div $(\vec{F}\times\vec{G}) = \vec{G} \bullet \overrightarrow{\text{curl}}\vec{F} - \vec{F} \bullet \overrightarrow{\text{curl}}\vec{G}$

⋆ 9. div $\overrightarrow{\text{curl}}\vec{F} = 0$ if $\vec{F} \in C^2$

10. $\overrightarrow{\text{curl}}(f\vec{F}) = f\overrightarrow{\text{curl}}\vec{F} + \vec{\nabla} f \times \vec{F}$

⋆ 11. $\overrightarrow{\text{curl}}\vec{\nabla} f = \vec{0}$ if $f \in C^2$

12. $\vec{\nabla}^2(fg) = f\vec{\nabla}^2 g + g\vec{\nabla}^2 f + 2(\vec{\nabla} f \bullet \vec{\nabla} g)$
13. div $(\vec{\nabla} f \times \vec{\nabla} g) = 0$ if $f, g \in C^2$
14. div $(f\vec{\nabla} g - g\vec{\nabla} f) = f\vec{\nabla}^2 g - g\vec{\nabla}^2 f$

Note: Euler's Theorem **3.1** $\Longrightarrow$ 9, 11, 13

4.4. Gradient Vector Field

Given: a vector field $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ where $\mathbb{D}^3 \subset \mathbb{R}^3$

a scalar field $f\colon \mathbb{D}^3 \to \mathbb{R}$ and $f \in C^1$

$\vec{\nabla} f = \vec{F}$

Then: $\vec{F}$ is called a gradient vector field

f is called the potential of $\vec{F}$.

Thm. If: $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ is a vector field

$\vec{F} \in C^1$

$\overrightarrow{\text{curl}}\vec{F} \neq \vec{0}$

Then: $\vec{F}$ is NOT a gradient vector field .

Why: by 4.3 #11 above .

4.5. Geometry of the Divergence   pg 284-5

4.5a. Given: a VELOCITY vector field $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ of a compressible fluid

a point $(x_0, y_0, z_0) \in \mathbb{D}^3$ and a small $\varepsilon > 0$

$P(0)$ is the cube (of the fluid) based at (x_0, y_0, z_0) with sides determined by $\varepsilon\vec{i}, \varepsilon\vec{j}, \varepsilon\vec{k}$

$P(t)$ is the (approx.) parallelepiped that $P(0)$ is carried to t time-units later

$\mathcal{V}(t)$ = volume of $P(t)$.

Then:

$$\text{div } \vec{F}(x_0, y_0, z_0) = \frac{1}{\mathcal{V}(0)} \frac{d\mathcal{V}}{dt}\Big|_{\left[\substack{t=0 \\ (x,y,z)=(x_0,y_0,z_0)}\right]} .$$

So: div $\vec{F}$ is the rate of change of the fluid's volume, per unit vol., under the flow of the fluid

div $\vec{F}(x_0, y_0, z_0) > 0 \Rightarrow$ the fluid is exanding at $(x_0, y_0, z_0) \Rightarrow (x_0, y_0, z_0)$ is a source

div $\vec{F}(x_0, y_0, z_0) < 0 \Rightarrow$ the fluid is compressing at $(x_0, y_0, z_0) \Rightarrow (x_0, y_0, z_0)$ is a sink

4.5b. Have: a fluid flowing through a region $\mathbb{D}^3$ in $\mathbb{R}^3$ with:

$\vec{v}$ - velocity vector field of the fluid $\left(\frac{\text{m}}{\text{sec}}\right)$

δ - mass density scalar field of the fluid $\left(\frac{\text{gr}}{\text{m}^3}\right)$

$\overrightarrow{F} \stackrel{\text{def}}{=} \delta\vec{v}$ - mass flow rate density of the fluid $\left(\frac{\text{gr}}{\text{sec}} \text{ per m}^2\right)$

a porous surface $\mathcal{S}$ sitting in $\mathbb{D}^3$:

through which the fluid flows without hindrance & with outward UNIT normal $\vec{n}$.

Def: **flux** of $\overrightarrow{F}$ across $\mathcal{S}$ $\stackrel{\text{def}}{=}$ net mass of fluid crossing $\mathcal{S}$ per unit time $\left(\frac{\text{gr}}{\text{sec}}\right)$

$$= \frac{\big([\text{mass of fluid coming out of } \mathcal{S}] - [\text{mass of fluid going into } \mathcal{S}]\big) \text{ during } \Delta t}{\Delta t}$$

Note: $\begin{bmatrix} \text{coming out of } \mathcal{S} \\ \text{going into } \mathcal{S} \end{bmatrix}$ means the fluid is flowing away from the $\begin{bmatrix} \text{positive} \\ \text{negative} \end{bmatrix}$ side of $\mathcal{S}$.

Note: if $\delta \equiv 1$, then flux = the net volume of fluid crossing $\mathcal{S}$ per unit time.

(i) If: $\mathcal{S}$ is a small FLAT patch and $\vec{v}$ and δ are constant on $\mathcal{S}$

then: flux across $\mathcal{S}$ $= \left(\overrightarrow{F} \bullet \vec{n}\right)$ (area of $\mathcal{S}$) .

(ii) If: $\mathcal{S} \equiv \mathcal{S}(\varepsilon)$ is a rectangular box with:

center P_0, sides $\|$ to the coordinate axes, and of volume ε

then: $\mathcal{S}(\varepsilon)$ is the disjoint union of 6 small FLAT patches as above and

$$\text{div } \vec{F}(P_0) = \lim_{\varepsilon \to 0} \frac{\text{flux of } \vec{F} \text{ across } \mathcal{S}(\varepsilon)}{\text{vol } \mathcal{S}(\varepsilon)} .$$

So: div $\vec{F}(P_0)$ is the flux of $\vec{F}$ across a small element of volume containing P_0, per unit volume.

4.6. Rotation pg 278-9

Given: a (slanted/flat) solid rigid body B that is welded to an axis L

the axis L is rotating at a constant angular speed

some point Q on B .

So as L rotates, Q follows a circular path in a plane $\perp$ to L .

Let: $\vec{v}$ = velocity vector field of B

$\vec{w}$ = angular velocity vector of B .

So:

- $\triangleright$ $\vec{w} \parallel L$
- $\triangleright$ $\|\vec{w}\| \equiv \omega \equiv$ angular speed of $B \equiv \frac{d(\text{angle})}{dt} = \frac{\text{speed of } Q}{d(Q,L)}$.

Then: $\vec{v} = \vec{w} \times \overrightarrow{OQ}$

$\overrightarrow{\text{curl}}\, \vec{v} = 2\vec{w}$.

4.7. Geometry of the Curl pg 279

Given: a VELOCITY vector field $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ of a fluid

a point $(x,y,z) \in \mathbb{D}^3$

a RIGID body B centered at (x,y,z) .

Then: $\overrightarrow{\text{curl}}\, \vec{F}(x,y,z)$ is twice the angular velocity vector of B

as it rotates about its own axis while in the fluid at (x,y,z) .

Def: $\overrightarrow{\text{curl}}\, \vec{F} = 0 \overset{\text{def}}{\iff} \vec{F}$ is IRROTATIONAL .

4.8. Key Idea

Given: a VELOCITY vector field $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ of a fluid .

Then: div $\vec{F}$ measures how much the fluid is expanding

$\overrightarrow{\text{curl}}\, \vec{F}$ measures how much an object is rotating about its own axis when placed in the fluid.

 a review

5.1. Single Integral

Given: a function $f\colon [a,b] \to \mathbb{R}$

a partition $P\colon a = x_0 < x_1 < \ldots < x_n = b$ of $[a,b]$

a selection $\{c_i\}_{i=1}^n$ where $c_i \in [x_{i-1}, x_i]$

Defs: $l\,(\,[x_{i-1},x_i]\,) \;=\;$ length $[x_{i-1},x_i]$

<u>mesh</u> of $P = |P| \;=\; \max_{1\leqslant i\leqslant n} l\,(\,[x_{i-1},x_i]\,)$

<u>Reimann sum</u> of $(f, P, \{c_i\}) = \sum_{i=1}^n \; f(c_i) \,\cdot\, l\,(\,[x_{i-1},x_i]\,)$

DEF: f is <u>integrable</u> over $[a,b]$ with $\int_{[a,b]} f(x)\,dx \;=\; I \;\overset{\text{def}}{\Longleftrightarrow}$

$\lim_{|P|\to 0} \sum_{i=1}^n \; f(c_i) \,\cdot\, l\,(\,[x_{i-1},x_i]\,) \;=\; I$ for each selection of $\{c_i\}$'s .

So: $\int_{[a,b]} 1\;dx \;=\; l\,(\,[a,b]\,)$

and: $\sum_{i=1}^n \; f(c_i) \,\cdot\, \frac{l(\,[x_{i-1},x_i]\,)}{l(\,[a,b]\,)} \;\approx\;$ avg. value of f over $[a,b] \;=\; \dfrac{\int_{[a,b]} f(x)\,dx}{l\,(\,[a,b]\,)}$.

Def: the <u>area</u> of the region btw. the graph of $y = f(x)$ and $[a,b] \;\overset{\text{def}}{=\!=}\; \int_{[a,b]} f(x)\,dx$.

Thm: If f is continuous on $[a,b]$, then f is integrable over $[a,b]$.

<u>Fundamental Theorem of Calculus</u>: If $f = F'$ is continuous, then $\int_{[a,b]} f(x)\,dx \;=\; F(b) - F(a)$.

5.2. Double Integral

Given: $R^2 \subset [a_x, b_x] \times [a_y, b_y] \subset \mathbb{R}^2$

a function $f\colon R^2 \to \mathbb{R}$

a partition $P_x\colon a_x = x_0 < x_1 < \ldots < x_n = b_x$ of $[a_x, b_x]$

a partition $P_y\colon a_y = y_0 < y_1 < \ldots < y_m = b_y$ of $[a_y, b_y]$.

Form: rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

Pick: a selection $\{c_{ij}\}$ where $c_{ij} \in R_{ij}$.

Defs: $A\,(R_{ij}) \;=\;$ area of R_{ij}

$\mathcal{R} = \{R_{ij} \;:\; R_{ij} \subset R^2\}$

<u>mesh</u> of $\mathcal{R} = |\mathcal{R}| \;=\; \max_{ij} A\,(R_{ij})$

<u>Reimann sum</u> of $(f, P_x, P_y, \{c_{ij}\}) = \sum_{R_{ij}\in\mathcal{R}} \; f(c_{ij}) \,\cdot\, A\,(R_{ij})$.

DEF: f is <u>integrable</u> over R^2 with $\iint_{R^2} f(x,y)\;dA \;=\; I \;\overset{\text{def}}{\Longleftrightarrow}$

$\lim_{|\mathcal{R}|\to 0} \sum_{R_{ij}\in\mathcal{R}} \; f(c_{ij}) \,\cdot\, A\,(R_{ij}) \;=\; I$ for each selection of $\{c_{ij}\}$'s .

So: $\iint_{R^2} 1\;dA \;=\;$ area of R^2

and: $\sum_{R_{ij}\in\mathcal{R}} \; f(c_{ij}) \,\cdot\, \frac{A(R_{ij})}{A(R^2)} \;\approx\;$ avg. value of f over $R^2 \;=\; \frac{\iint_{R^2} f(x,y)\;dA}{\text{area of } R^2}$.

Def: the <u>volume</u> of the solid btw. the graph of $z = f(x,y)$ and $R^2 \;\overset{\text{def}}{=\!=}\; \iint_{R^2} f(x,y)\;dA$.

Def: R^2 is elementary if, for continuous functions g_i, either:

R^2 is of type 1 = type_y $\overset{\text{def}}{\iff}$ $R^2 = \{(x,y) \in \mathbb{R}^2 : g_1(x) \leqslant y \leqslant g_2(x) \text{ and } a \leqslant x \leqslant b\}$

$\leftrightsquigarrow$ draw typical rectangle $\parallel$ to y-axis

R^2 is of type 2 = type_x $\overset{\text{def}}{\iff}$ $R^2 = \{(x,y) \in \mathbb{R}^2 : g_1(y) \leqslant x \leqslant g_2(y) \text{ and } a \leqslant y \leqslant b\}$

$\leftrightsquigarrow$ draw typical rectangle $\parallel$ to x-axis

R^2 is of type 3 = type_{xy} $\overset{\text{def}}{\iff}$ R^2 is of both type_x and type_y .

If: $f\colon R^2 \to \mathbb{R}$ is continuous and R^2 is elementary,

then: $\iint_{R^2} f(x,y)\, dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\right] dx$ if R^2 is of type_y

and: $\iint_{R^2} f(x,y)\, dA = \int_a^b \left[\int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\right] dy$ if R^2 is of type_x .

5.3. Triple Integral

Given: $R^3 \subset [a_x, b_x] \times [a_y, b_y] \times [a_z, b_z] \subset \mathbb{R}^3$

a function $f\colon R^3 \to \mathbb{R}$

a partition $P_x\colon a_x = x_0 < x_1 < \ldots < x_n = b_x$ of $[a_x, b_x]$

a partition $P_y\colon a_y = y_0 < y_1 < \ldots < y_m = b_y$ of $[a_y, b_y]$

a partition $P_z\colon a_z = z_0 < z_1 < \ldots < z_l = b_z$ of $[a_z, b_z]$.

Form: rectangular cubes (ie. boxes) $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$.

Pick: a selection $\{c_{ijk}\}$ where $c_{ijk} \in B_{ijk}$.

Defs: $V(B_{ijk})$ = volume of B_{ijk}

$\mathcal{V} = \{B_{ijk} : B_{ijk} \subset R^3\}$

mesh of $\mathcal{V} = |\mathcal{V}| = \max_{ijk} V(B_{ijk})$

Reimann sum of $(f, P_x, P_y, P_z, \{c_{ijk}\}) = \sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot V(B_{ijk})$.

DEF: f is integrable over R^3 with $\iiint_{R^3} f(x,y,z)\, dV = I$ $\overset{\text{def}}{\iff}$

$\lim_{|\mathcal{V}| \to 0} \sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot V(B_{ijk}) = I$ for each selection of $\{c_{ijk}\}$'s .

So: $\iiint_{R^3} 1\, dV$ = volume of R^3

and: $\sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot \frac{V(B_{ijk})}{V(R^3)} \approx$ avg. value of f over $R^3 = \frac{\iiint_{R^3} f(x,y,z)\, dV}{V(R^3)}$

Def: R^3 is z-simple if, $R^3 = \{(x,y,z) \in \mathbb{R}^3 : g_1(x,y) \leqslant z \leqslant g_2(x,y) \text{ and } (x,y) \in R^2\}$ where :

▷ $g_i \in C^1$, ie. g_i have continuous 1st-order partial derivatives

▷ R^2 is elementary , as in **5.2** .

$\rightsquigarrow$ draw typical box with height $\parallel$ to z-axis

If: f is continuous and R^3 is z-simple,

then: $\iiint_{R^3} f(x,y,z)\, dV = \iint_{R^2} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz\right] dA$.

Rmk: Analogs hold for x-simple and y-simple.

 new stuff

6.1. Recall the 1D Change of Variables Formula:

Have: $I^* \xrightarrow{g} I \xrightarrow{f} \mathbb{R}$

where: $I^* = [a,b] \subset \mathbb{R}$

$g \in C^1(I^*)$, ie. g' is continuous on I^*

$I = g(I^*) = [c,d]$

f is continuous .

Then: $$\underbrace{\int_{g(a)}^{g(b)} f(x)\,dx}_{\text{get something easy to compute}} \quad \underbrace{\overset{(*)}{=}}_{\text{so find } g \text{ and } I^* \hookleftarrow} \quad \underbrace{\int_a^b f(g(u))\,\frac{dg}{du}\,du}_{\text{given but hard to compute } \hookleftarrow} .$$

usually:

If: $\frac{dg}{du} \geqslant 0$ on I^* (and so $I = [g(a), g(b)]$)

or: $\frac{dg}{du} \leqslant 0$ on I^* (and so $I = [g(b), g(a)]$)

Then: formula $(*)$ becomes $\int_I f(x)\,dx \overset{(**)}{=} \int_{I^*} f(g(u)) \left|\frac{dg}{du}\right| du$.

i.e.: if you prefer, let $g(u) = x(u)$ so $\int_I f(x)\,dx \overset{(**)}{=} \int_{I^*} f(x(u)) \left|\frac{dx}{du}\right| du$.

Goal the 2D & 3D Change of Variables Formulas:

Have: $D^* \xrightarrow{T} D \xrightarrow{f} \mathbb{R}$

$D^*, D \subset \mathbb{R}^{2 \text{ or } 3}$

$T(D^*) = D$

D^*, D, T, f are *nice* .

Want: an anolog to $(**)$,

like: $\iint_D f(x,y)\,dA \overset{(2D)}{=} \iint_{D^*} f(T(u,v))\,\boxed{?}\,dA$

or: $\iiint_D f(x,y,z)\,dV \overset{(3D)}{=} \iiint_{D^*} f(T(u,v,w))\,\boxed{?}\,dV$.

6.2. Interior & Boundary cf. **5.2** & **5.3**

For: the elementary region $R^2 = \{(x,y) \in \mathbb{R}^2 : a \leqslant x \leqslant b \text{ and } g_1(x) \leqslant y \leqslant g_2(x)\}$

the interior of R^2 is $\{(x,y) \in \mathbb{R}^2 : a < x < b \text{ and } g_1(x) < y < g_2(x)\}$

the boundary of R^2 is $R^2 \setminus [\text{the interior of } R^2]$.

For: the z-simple region $R^3 = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2 \text{ and } g_1(x,y) \leqslant z \leqslant g_2(x,y)\}$

the interior of R^3 is $\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in [\text{the interior of } R^2] \;\&\; g_1(x,y) < z < g_2(x,y)\}$

the boundary of R^3 is $R^3 \setminus [\text{the interior of } R^3]$.

6.3. One-to-One & Onto & Jacobian

Have: $T \colon D^* \to D$

$D^*, D \subset \mathbb{R}^n$

Def: T is one-to-one $\Leftrightarrow$ if $\vec{a}_i \in D^*$ and $\vec{a}_1 \neq \vec{a}_2$ then $T(\vec{a}_1) \neq T(\vec{a}_2)$

$\Leftrightarrow$ if $\vec{a}_i \in D^*$ and $T(\vec{a}_1) = T(\vec{a}_2)$ then $\vec{a}_1 = \vec{a}_2$

$\Leftrightarrow$ T does NOT send two different points of D^* to the same point in D

$\Leftrightarrow$ T does NOT fold-up D

$\overset{n=1}{\Leftrightarrow}$ T passes the hortizontal line test.

Rmk: in **6.1**, if g is 1-to-1, then either $\frac{dg}{du} \geqslant 0$ or $\frac{dg}{du} \leqslant 0$ on I^* .

Def: T is onto $\Leftrightarrow$ $T(D^*) = D$

$\Leftrightarrow$ if $\vec{b} \in D$ then there is $\vec{a} \in D^*$ so that $T(\vec{a}) = \vec{b}$

$\Leftrightarrow$ T maps D^* onto *all* of D .

If: $T \in C^1(D^*)$

then: Jacobian matrix of T $= J_T = D_{_M}T$ (recall $D_{_M}$ was defined in **2.2**)

and: Jacobian of T $= \det J_T$.

So : for $n = 2$,

$$T(u,v) = \langle\, x(u,v)\,,\, y(u,v)\,\rangle$$
$$\det J_T = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and for $n = 3$,

$$T(u,v,w) = \langle\, x(u,v,w)\,,\, y(u,v,w)\,,\, z(u,v,w)\,\rangle$$
$$\det J_T = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} .$$

To change to cylindrical coordinates : (see **1.8**)

let: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

so: $T(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$

and: $\det J_T = r$ and $|\det J_T| = r$

To change to spherical coordinates : (see **1.9**)

let: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

so: $T(\rho,\theta,\phi) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$

and: $\det J_T = -\rho^2\sin\phi$ and $|\det J_T| = \rho^2\sin\phi$

6.4. Examples in $\mathbb{R}^2$: $T\colon D^* \to D$

6.4.1. Polar Coordinates:

▷. $T(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle \quad\Rightarrow\quad J_T = \_\_\_\_\_\_\_\_ \quad\Rightarrow\quad \det J_T = \_\_\_\_\_$

a. $D^* = [a,b]\times[0,\pi/2]$ where $0 < a < b \quad\Rightarrow\quad T(D^*) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

b. $D^* = [a,b]\times[0,3\pi]$ where $0 < a < b \quad\Rightarrow\quad T(D^*) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

c. $D^* = [0,b]\times[0,\theta_0]$ where $0 < b$ and $0 < \theta_0 < 2\pi \quad\Rightarrow\quad T(D^*) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

6.4.2. Linear Transformations: here $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

▷. $T(u,v) = \mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix} + \vec{b} = \begin{bmatrix} a_{11}u + a_{12}v + b_1 \\ a_{21}u + a_{22}v + b_2 \end{bmatrix} = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$

Facts from linear algebra:

(i) $\det\mathbf{A} \neq 0 \quad\Leftrightarrow\quad T$ is 1-to-1 (see § 6.1 Ex. 8)

(ii) $\det\mathbf{A} \neq 0 \quad\Leftrightarrow\quad T(\mathbb{R}^2) = \mathbb{R}^2$ (see § 6.1 Ex. 9)

(iii) $\mathbf{A}\vec{d_1} \times \mathbf{A}\vec{d_2} = (\det\mathbf{A})\,(\vec{d_1}\times\vec{d_2})$ (just write it out)

(iv) $J_T = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$.

Consider parallelograms: here $\vec{d_i} \neq \vec{0}$

Let: $D^* = \{\vec{p} + \lambda\vec{d_1} + \mu\vec{d_2} : 0 \leqslant \lambda,\mu \leqslant 1\}$

so: D^* is a parallelogram $\Leftrightarrow$ $\vec{d_1} \nparallel \vec{d_2}$ $\Leftrightarrow$ $\vec{d_1}\times\vec{d_2} \neq \vec{0}$.

Note: $T(D^*) = \{(\mathbf{A}\vec{p} + \vec{b}) + \lambda\,\mathbf{A}\vec{d_1} + \mu\,\mathbf{A}\vec{d_2} : 0 \leqslant \lambda,\mu \leqslant 1\}$.

So: $T(D^*)$ is a parallelogram $\Leftrightarrow$ D^* is a parallelogram and $\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$.

Note: $\mathbf{A}\vec{i} = \langle\, a_{11}\,,\ a_{21}\,\rangle$ and $\mathbf{A}\vec{j} = \langle\, a_{12}\,,\ a_{22}\,\rangle$,

★. so: $T\left([0,1]\times[0,1]\right) = \left\{\vec{b} + \lambda\,\langle\, a_{11}\,,\ a_{21}\,\rangle + \mu\,\langle\, a_{12}\,,\ a_{22}\,\rangle : 0 \leqslant \lambda,\mu \leqslant 1\right\}$.

If: D^* & $T(D^*)$ are parallelograms, then

$$\text{area } T(D^*) = \left\|(\,\mathbf{A}\vec{d_1}\,)\times(\,\mathbf{A}\vec{d_2}\,)\right\| \overset{\text{(iii)}}{=} |\det\mathbf{A}|\,\left\|\vec{d_1}\times\vec{d_2}\right\| = |\det\mathbf{A}|\ \text{area }(D^*) \overset{\text{(iv)}}{=} |\det J_T|\ \text{area }(D^*)\,.$$

6.5. Change of Variables Formula

Let: $D^* \xrightarrow{T} D \xrightarrow{f} \mathbb{R}$

$D^*, D \subset \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R}^3 \end{bmatrix}$ and D^* is $\begin{bmatrix} \text{elementary} \\ \text{simple} \end{bmatrix}$ (see $\begin{bmatrix} \mathbf{5.2} \\ \mathbf{5.3} \end{bmatrix}$)

$T \in C'(D^*)$

T is 1-to-1 on the interior of D^*

$T(D^*) = D$

f is continuous .

For: 2D-case:

$$\iint_D f(x,y)\, dA \quad = \quad \iint_{D^*} f(T(u,v))\ |\det J_T|\ dA$$

mnemonically:

$$\iint_D f(x,y) \text{ ``}dxdy\text{''} \quad = \quad \iint_{D^*} f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \text{ ``}dudv\text{''} \ .$$

For: 3D-case:

$$\iiint_D f(x,y,z)\, dV \quad = \quad \iiint_{D^*} f(T(u,v,w))\ |\det J_T|\ dV$$

mnemonically:

$$\iiint_D f(x,y,z) \text{ ``}dxdydz\text{''} \quad = \quad \iiint_{D^*} f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \text{ ``}dudvdw\text{''} \ .$$

7.h. Ch 7 – Summary

7.h.1. Given: a path $[a,b] \xrightarrow{\vec{c}} \mathbb{R}^n$

a surface $\underset{\mathbb{R}^2}{\underset{\cap}{D}}{}^2 \xrightarrow[\text{onto}]{\Phi} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$

a scalar field $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

a vector field $\mathbb{R}^n \xrightarrow{\vec{F}} \mathbb{R}^n$

with: everything nice & continuous.

7.b Path Integral — averages f along the path $\vec{c}$

$\int_{\vec{c}} f\, ds = \int_a^b f(\vec{c}(t))\, \|\vec{c}\,'(t)\|\, dt$

7.c Line Integral — work done by force $\vec{F}$ on a puffo as he moves along the $\vec{c}$

$\int_{\vec{c}} \vec{F} \bullet d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \bullet \vec{c}\,'(t)\, dt$

7.e Surface Integral — averages f over the surface $\mathcal{S}$

$\iint_{\mathcal{S}} f\, dS = \iint_{D^2} f(\Phi(u,v))\, \left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|\, dudv$

7.g Surface Integral — the flux of $\vec{F}$ across the surface $\mathcal{S}$

$\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} = \iint_{D^2} \vec{F} \bullet \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right)\, dudv$

$= \iint_{\mathcal{S}} \left[\vec{F} \bullet \vec{n}\right]\, dS$

Note: path & line integrals change integrals over paths to single integrals (as in **5.1**)

surface integrals change integrals over surfaces to double integrals (as in **5.2**) .

7.h.2. Integrals over paths

$\vec{c}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{c}\,'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$ds = \|\vec{c}\,'\|\, dt$

$d\vec{s} = \vec{c}\,'\, dt$

7.h.3. Integrals over Surfaces

$\Phi(u,v) = (\, x\,(u,v)\,, y\,(u,v)\,, z\,(u,v)\,)$

$\overrightarrow{T_u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle$

$\overrightarrow{T_v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \rangle$

$dS = \left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\| dudv = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}\, dudv$ where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$\underset{\mathbf{7.g}}{\underset{\uparrow}{d\vec{S}}} = \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right) dudv = \vec{n}\, \underset{\mathbf{7.e}}{\underset{\uparrow}{dS}}$ where $\vec{n}$ is the outward unit normal vector to $\mathcal{S}$

7.a. Paths/Curves

▷ Review **2.6**

▷ Given: $[a_1, b_1] \xrightarrow[\text{onto}]{h} [a,b] \xrightarrow{\vec{c}} \mathbb{R}^n$

with: $h \in C^1$ and h is 1-to-1 on $[a_1, b_1]$

$\vec{c}$ is piecewise C^1 on $[a,b]$

then: $\vec{p} \stackrel{\text{def}}{=} \vec{c} \circ h \colon [a_1, b_1] \to \mathbb{R}^n$ is a REPARAMETERIZATION of $\vec{c}$.

Note: $\{\vec{c}(t) \colon a \leqslant t \leqslant b\} = \{\vec{p}(t) \colon a_1 \leqslant t \leqslant b_1\} = \mathcal{C}$.

Since: h is 1-to-1, there are 2 possibilities:

either: $h' > 0$ on $[a_1, b_1] \Rightarrow [a,b] = [h(a_1), h(b_1)] \Rightarrow h$ PRESERVES ORIENTATION

or: $h' < 0$ on $[a_1, b_1] \Rightarrow [a,b] = [h(b_1), h(a_1)] \Rightarrow h$ REVERSES ORIENTATION .

Since: $\vec{p}\,'(t) = \vec{c}\,'(h(t))\, h'(t)$

one thinks of h as changing the speed at which a point moves along the curve $\mathcal{C}$.

Ex: If: $[a,b] \xrightarrow[\text{onto}]{h} [a,b] \xrightarrow{\vec{c}} \mathbb{R}^n$

with: $h(t) = a + b - t$

then: $\vec{p}(t) = \vec{c}(a+b-t) = \vec{c}_{\text{op}}(t) =$ the opposite path to $\vec{c}$

$=$ a path in the opposite direction as $\vec{c}$

but with the same speed as $\vec{c}$.

Ex: If: $[0,1] \xrightarrow[\text{onto}]{h} [a,b] \xrightarrow{\vec{c}} \mathbb{R}^n$

with: $h(t) = a + (b-a)\,t$

then: $\vec{p}(t) = \vec{c}(a + (b-a)\,t)$

$=$ a path in the same direction as $\vec{c}$

but with the speed of (b-a) times the speed of $\vec{c}$.

7.b. Paths Integral

7.b.1. Given: $[a,b] \xrightarrow{\vec{c}} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$

with: $\vec{c} \in C^1[a,b]$

f is a scalar field

f is continuous on $\vec{c}\,([a,b])$

then: the PATH INTEGRAL of f over $\vec{c}$ is $\boxed{\int_{\vec{c}} f\,ds \overset{\text{def}}{=\!=} \int_a^b f(\vec{c}(t))\ \|\vec{c}\,'(t)\|\ dt}$.

If: $\vec{c}$ is piecewise C^1 on $[a,b]$

then: we cope in the obvious way.

7.b.2. Key Idea:

Form: a partition of $[a,b]$: $a = t_0 < t_1 < \ldots < t_n = b$,

with: $\Delta t_i \equiv t_i - t_{i-1} \approx 0$,

pick: a selection $t_i^* \in [t_{i-1}, t_i]$.

Note: $\vec{c}\,'(t_i^*) \approx [\ \vec{c}(t_i) - \vec{c}(t_{i-1})\ ]/\Delta t_i$.

Then:

$$\begin{aligned}
\int_{\vec{c}} f\,ds &= \int_a^b [\ f(\vec{c}(t))\ \|\vec{c}\,'(t)\|\ ]\ dt \\
&\approx \sum_{i=1}^n [\ f(\vec{c}(t_i^*))\ \|\vec{c}\,'(t_i^*)\|\ ]\ \Delta t_i \\
&= \sum_{i=1}^n f(\vec{c}(t_i^*))\ [\ \|\vec{c}\,'(t_i^*)\|\ \Delta t_i\ ] \\
&\approx \sum_{i=1}^n f(\vec{c}(t_i^*))\ \|\ \vec{c}(t_i) - \vec{c}(t_{i-1})\ \| \\
&\approx \sum_{i=1}^n f(\vec{c}(t_i^*))\ [\text{length of the curve } \vec{c}\,([t_{i-1}, t_i])\,]\ .
\end{aligned}$$

7.b.3. Examples:

▷ $\int_{\vec{c}} 1\,ds$ = length of the path $\vec{c}$.

▷ If: $\vec{c}$ represents a wire in $\mathbb{R}^3$

with: $\rho \colon \mathbb{R}^3 \to \mathbb{R}$ the mass density (eg. grams/cm) of $\vec{c}$

then: the total mass of the wire $= \int_{\vec{c}} \rho\,ds$

and: the center of mass of the wire $= \left(\frac{\int_{\vec{c}} x\rho\,ds}{\int_{\vec{c}} \rho\,ds}\ ,\ \frac{\int_{\vec{c}} y\rho\,ds}{\int_{\vec{c}} \rho\,ds}\ ,\ \frac{\int_{\vec{c}} z\rho\,ds}{\int_{\vec{c}} \rho\,ds} \right)$.

▷ the AVERAGE VALUE of f along $\vec{c}$ is $\frac{\int_{\vec{c}} f\,ds}{\int_{\vec{c}} 1\,ds}$

eg: $f(x,y,z)$ is the temperature at the point (x,y,z)

so: average value of f along $\vec{c}$ = average tempature of f along $\vec{c}$.

▷ If: $[a,b] \xrightarrow{\vec{c}} \mathbb{R}^{\mathbf{2}} \xrightarrow{f} [0,\infty)$ and so $\vec{c}\,([a,b])$ sits in the xy-plane

then: $\int_{\vec{c}} f\,ds$ is the surface area of one side of the fence over $\vec{c}$ with height given by f .

7.b.4. Given: **7.b.1** setting

a reparameterization $\vec{p}$ of $\vec{c}$

then: $\int_{\vec{p}} f\,ds = \int_{\vec{c}} f\,ds$.

7.c. Line Integral

7.c.1. Given: $[a,b] \xrightarrow{\vec{c}} \mathbb{R}^n \xrightarrow{\vec{F}} \mathbb{R}^n$

with: $\vec{c} \in C^1[a,b]$

$\vec{F}$ is a vector field

$\vec{F}$ is continuous on $\vec{c}\,([a,b])$

then: the LINE INTEGRAL of $\vec{F}$ over $\vec{c}$ is $\boxed{\int_{\vec{c}} \vec{F} \bullet d\vec{s} \stackrel{\text{def}}{=} \int_a^b \vec{F}(\vec{c}(t)) \bullet \vec{c}\,'(t)\; dt}$.

7.c.2. Key Idea: notation as in **7.b.2** and setting as in **7.c.1** :

Then:
$$\begin{aligned}\int_{\vec{c}} \vec{F} \bullet d\vec{s} &= \int_a^b \left[\vec{F}(\vec{c}(t)) \bullet \vec{c}\,'(t)\right] dt \\ &\approx \sum_{i=1}^n \left[\vec{F}(\vec{c}(t_i^*)) \bullet \vec{c}\,'(t_i^*)\right] \Delta t_i \\ &\approx \sum_{i=1}^n \vec{F}(\vec{c}(t_i^*)) \bullet \left[\vec{c}\,'(t_i^*)\,\Delta t_i\right] \\ &\approx \sum_{i=1}^n \vec{F}(\vec{c}(t_i^*)) \bullet \left[\vec{c}\,(t_i) - \vec{c}\,(t_{i-1})\right] .\end{aligned}$$

7.c.3. Ex. if: $\vec{F}$ is a force field acting on a puffo

then: $\int_{\vec{c}} \vec{F} \bullet d\vec{s}$ is the work done by $\vec{F}$ on the puffo as he moves along the path $\vec{c}$.

7.c.4. Differential Form Notation:

If: $\vec{c}(t) = \langle x(t), y(t), z(t)\rangle$ and $\vec{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$

then:
$$\int_{\vec{c}} \vec{F} \bullet d\vec{s} = \int_a^b \langle F_1, F_2, F_3\rangle \bullet \langle \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt}\rangle\; dt = \int_a^b \left[F_1 \tfrac{dx}{dt} + F_2 \tfrac{dy}{dt} + F_3 \tfrac{dz}{dt}\right] dt$$
$$\stackrel{\text{notation}}{=} \int_{\vec{c}} F_1\, dx + F_2\, dy + F_3\, dz .$$

7.c.5. From a Line Integral (**7.c.1** setting) to a Path Integral (**7.b.1** setting) .

If: $\vec{c}\,'(t) \neq \vec{0}$ on $[a,b]$ and $\vec{c}$ is one-to-one on $[a.b]$

so: $\vec{T}(t) \stackrel{\text{def}}{=} \vec{c}\,'(t) \,/\, \|\vec{c}\,'(t)\|$ is the unit tangent vector to $\vec{c}(t)$

then:
$$\begin{aligned}\int_{\vec{c}} \vec{F} \bullet d\vec{s} &= \int_a^b \left[\vec{F}(\vec{c}(t)) \bullet \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\right] \|\vec{c}\,'(t)\|\; dt = \int_a^b \left[\vec{F}(\vec{c}(t)) \bullet \vec{T}(t)\right] \|\vec{c}\,'(t)\|\; dt \\ &= \int_a^b \left[\vec{F}(\vec{c}(t)) \bullet \vec{T}\left(\vec{c}^{\,-1}(\vec{c}\,(t))\right)\right] \|\vec{c}\,'(t)\|\; dt \\ &= \int_{\vec{c}} \left[\vec{F} \bullet \left(\vec{T} \circ \vec{c}^{\,-1}\right)\right] ds\end{aligned}$$

$\rightsquigarrow$ think of $\vec{F} \bullet \left(\vec{T} \circ \vec{c}^{\,-1}\right)$ as the tangential component of $\vec{F}$ along $\vec{c}$.

7.c.6. Given: the **7.c.1** setting and a reparameterization $\vec{p}$ of $\vec{c}$.

If $\vec{p}$ preserves orientation then $\int_{\vec{p}} \vec{F} \bullet d\vec{s} = {}^{+}\int_{\vec{c}} \vec{F} \bullet d\vec{s}$.

If $\vec{p}$ reverses orientation then $\int_{\vec{p}} \vec{F} \bullet d\vec{s} = {}^{-}\int_{\vec{c}} \vec{F} \bullet d\vec{s}$.

7.c.7. Given: $\vec{c}\colon [a,b] \to \mathbb{R}^3$ with $\vec{c}$ piecewise C^1

$f\colon \mathbb{R}^3 \to \mathbb{R}$ with $f \in C^2$

so $\vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle \colon \mathbb{R}^3 \to \mathbb{R}^3$ and $\vec{\nabla} f \in C^1$

then: $\int_{\vec{c}} \vec{\nabla} f \bullet d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$.

7.d. Surfaces

7.d.1. Given: $\Phi\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$

so: $\Phi(u,v) = (\, x\,(u,v)\,, y\,(u,v)\,, z\,(u,v)\,)$.

Def: Φ is a PARAMETRIZED SURFACE

$\mathcal{S}$ is the SURFACE corresponding to Φ .

If Φ is differentiable (resp. C^1) at $(u,v) \in D^2$

ie. x and y and z are each differentiable (resp. C^1) at $(u,v) \in D^2$

then $\mathcal{S}$ is a differentiable (resp. C^1) surface at $\Phi(u,v) \in \mathcal{S}$.

If $\mathcal{S}$ is a differentiable (resp. C^1) at each $\Phi(u,v) \in \mathcal{S}$

then $\mathcal{S}$ is a differentiable (resp. C^1) surface .

Rmk: Think of Φ as twisting and bending the 2D region D^2 into a surface $\mathcal{S}$ sitting in 3D.

7.d.2. Key Ideas and Defs:

Setup Given: $\Phi\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$ a parametrized surface

$(u_0, v_0) \in D^2$ with Φ differentiable at (u_0, v_0) ,

so $\Phi(u_0, v_0) \equiv (x_0, y_0, z_0) \in \mathcal{S}$.

(A) Then $\vec{c}_{u_0}(t) = \Phi(u_0, t)$ is a path whose image lies on $\mathcal{S}$ and goes through (x_0, y_0, z_0) at $t = v_0$

and a tangent vector to $\vec{c}_{u_0}$ at (x_0, y_0, z_0) is: $\overrightarrow{T_v}(u_0, v_0) \;\equiv\; \vec{c}\,'_{u_0}(v_0) \;=\; \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \rangle\big|_{(u_0,v_0)}$

so $\overrightarrow{T_v}(u_0, v_0)$ is the instantaneous rate of change in Φ, along $\vec{c}_{u_0}(t)$, at $\Phi(u_0, v_0)$.

Lkws $\vec{c}_{v_0}(t) = \Phi(t, v_0)$ is a path whose image lies on $\mathcal{S}$ and goes through (x_0, y_0, z_0) at $t = u_0$

and a tangent vector to $\vec{c}_{v_0}$ at (x_0, y_0, z_0) is: $\overrightarrow{T_u}(u_0, v_0) \;=\; \vec{c}\,'_{v_0}(u_0) \;=\; \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle\big|_{(u_0,v_0)}$

so $\overrightarrow{T_u}(u_0, v_0)$ is the instantaneous rate of change in Φ, along $\vec{c}_{v_0}(t)$, at $\Phi(u_0, v_0)$.

⇝ Note

$$\left[\overrightarrow{T_u} \times \overrightarrow{T_v}\right](u_0, v_0) \;=\; \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}_{(u_0,v_0)} .$$

(B) If $\left[\overrightarrow{T_u} \times \overrightarrow{T_v}\right](u_0, v_0) \neq \vec{0}$

then $\left[\overrightarrow{T_u} \times \overrightarrow{T_v}\right](u_0, v_0)$ is NORMAL to the surface $\mathcal{S}$ at $\Phi(u_0, v_0)$

so an equation of the TANGENT PLANE to the suface $\mathcal{S}$ at the point (x_0, y_0, z_0) is:

$$< x - x_0, y - y_0, z - z_0 > \;\bullet\; \left[\overrightarrow{T_u} \times \overrightarrow{T_v}\right](u_0, v_0) \;=\; 0 .$$

(C) Let: $\left[\overrightarrow{T_u} \times \overrightarrow{T_v}\right](u_0, v_0) \neq \vec{0}$

ΔA_0 be the rectangle at (u_0, v_0), with sides $(\Delta u)\vec{i}$ and $(\Delta v)\vec{j}$.

So $\Phi(\Delta A_0)$ is approximately the parallelogram at $\Phi(u_0, v_0)$ with sides:

$$\Phi(u_0, v_0 + \Delta v) - \Phi(u_0, v_0) = \vec{c}_{u_0}(v_0 + \Delta v) - \vec{c}_{u_0}(v_0) \approx \Delta v\, \overrightarrow{T_v}(u_0, v_0)$$

$$\Phi(u_0 + \Delta u, v_0) - \Phi(u_0, v_0) = \vec{c}_{v_0}(u_0 + \Delta u) - \vec{c}_{v_0}(u_0) \approx \Delta u\, \overrightarrow{T_u}(u_0, v_0)$$

so

$$\text{area } \Delta A_0 = (\Delta u)\ (\Delta v)$$

$$\text{surface area } \Phi(\Delta A_0) \approx \|\, T_u(u_0, v_0) \times T_v(u_0, v_0) \,\|\ (\Delta u)\ (\Delta v)$$

$$= \|\, T_u(u_0, v_0) \times T_v(u_0, v_0) \,\|\ (\text{ area of } \Delta A_0 \ ) \ .$$

(D) If $\left[\overrightarrow{T_u} \times \overrightarrow{T_v}\right](u_0, v_0) \neq \vec{0}$

then $\mathcal{S}$ is SMOOTH AT $\Phi(u_0, v_0) \in \mathcal{S}$.

Also $\mathcal{S}$ is SMOOTH if $\mathcal{S}$ is smooth at each point of $\mathcal{S}$.

Rmk: loosely speaking, a smooth surfaces has no corners or breaks.

7.d.3. Have $\Phi\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$ a parameterized surface

with D^2 is an elementary region in uv-plane

Φ is C^1 and 1-to-1 on the *interior of* D^2

$\mathcal{S}$ is smooth, except possibly at a finite number of points of $\mathcal{S}$

then Φ & $\mathcal{S}$ are <u>NICE</u> .

If Φ & $\mathcal{S}$ are a (finite, disjoint) union of NICE Φ_i's & $\mathcal{S}_i$'s

then Φ & $\mathcal{S}$ are <u>PIECEWISE NICE</u> (pw-nice) .

7.d.4. Have $\Phi\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$ a differentiable parameterized surface,

then

$$\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\| = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \ .$$

7.d.5. <u>Ex: Special Case</u>

Given $g\colon \underset{\mathbb{R}^2}{\underset{\cap}{D}}{}^2 \longrightarrow \mathbb{R}$.

Form: $\Phi\colon D^2 \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$ via $\Phi(x,y) = (x, y, g(x,y))$.

If: D^2 elementary

$g \in C^1$ on interior of D^2

then: Φ is nice

and:

$$\begin{aligned}
\overrightarrow{T_x} &= \langle\ 1, 0, \frac{\partial g}{\partial x}\ \rangle \\
\overrightarrow{T_y} &= \langle\ 0, 1, \frac{\partial g}{\partial y}\ \rangle \\
\overrightarrow{T_x} \times \overrightarrow{T_y} &= \langle\ -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\ \rangle \\
\left\| \overrightarrow{T_x} \times \overrightarrow{T_y} \right\| &= \sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2 + 1}
\end{aligned}$$

and if: $(x_0, y_0) \in D^2$

$P_0 \equiv (x_0, y_0, g(x_0, y_0)) \in \mathcal{S}$

$\vec{m}$ is ANY normal vector to $\mathcal{S}$ at P_0,

then:

$$\left\| \overrightarrow{T_x} \times \overrightarrow{T_y} \right\|_{P_0} = \frac{1}{\left| \cos \measuredangle \vec{k}\vec{m} \right|} = \frac{\|\vec{m}\|}{\left| \vec{k} \bullet \vec{m} \right|} .$$

7.f. Oriented Surfaces

7.f.1. Consider a parametrized surface $\Phi\colon \underset{\mathbb{R}^2}{\overset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\overset{\cap}{\mathcal{S}}}$

Intuitively, $\mathcal{S}$ is an ORIENTED SURFACE if it has 2 sides:

- ▷ one side is the outside, or POSITIVE SIDE
- ▷ the other side is the inside, or NEGATIVE SIDE .

At each point $\Phi(u,v) \in \mathcal{S}$, there is a (unique) OUTWARD unit normal vector $\vec{n}_{uv}$ with

- ▷ $\vec{n}_{uv}$ pointing away from the positive side of $\mathcal{S}$
- ▷ $-\vec{n}_{uv}$ pointing away from the negative side of $\mathcal{S}$.

We say that $\{\vec{n}_{uv}\}_{(u,v)\in D^2}$ orient $\mathcal{S}$.

If for each $\Phi(u_0,v_0) \in \mathcal{S}$ where $\mathcal{S}$ is smooth:

- ▷ $\left.\dfrac{\overrightarrow{T_u} \times \overrightarrow{T_v}}{\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|}\right|_{(u_0,v_0)} = \vec{n}_{u_0\,v_0}$, then Φ is ORIENTATION-PRESERVING
- ▷ $\left.\dfrac{\overrightarrow{T_u} \times \overrightarrow{T_v}}{\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|}\right|_{(u_0,v_0)} = -\vec{n}_{u_0\,v_0}$, then Φ is ORIENTATION-REVERSING .

So any parametrized surface for which $\overrightarrow{T_u} \times \overrightarrow{T_v}$ never vanishes can be considered as an oriented surface with a positive side determined by the direction of the $\overrightarrow{T_u} \times \overrightarrow{T_v}$'s.

7.f.2. Ex: Sphere ($x^2+y^2+z^2=\rho^2$)

$\Phi(\theta,\phi) = (\,\rho\sin\phi\cos\theta\,,\ \rho\sin\phi\sin\theta\,,\ \rho\cos\phi\,)$

$D^2 = \{(\theta,\phi)\colon 0 \leqslant \theta \leqslant 2\pi\ ,\ 0 \leqslant \phi \leqslant \pi\}$

$\overrightarrow{T_\theta} \times \overrightarrow{T_\phi} = -\,\langle \rho^2\sin^2\phi\cos\theta\,,\ \rho^2\sin^2\phi\sin\theta\,,\ \rho^2\sin\phi\cos\phi\rangle = -\,[\rho\ \sin\phi]\,\vec{\Phi}(\theta,\phi)$

$\left\|\overrightarrow{T_\theta} \times \overrightarrow{T_\phi}\right\| = \rho^2\ \sin\phi$

Φ is orientation-reversing

7.f.3. Ex: Special Case (see **7.d.5** and compare with **2.11**)

$\Phi(x,y) = (x,y,g(x,y))$ where g is differentiable

$$\left[\overrightarrow{T_x} \times \overrightarrow{T_y}\right](x_0,y_0) = \left.\frac{\langle -\frac{\partial g}{\partial x}\,,\ -\frac{\partial g}{\partial y}\,,\ 1\rangle}{\left\|\langle -\frac{\partial g}{\partial x}\,,\ -\frac{\partial g}{\partial y}\,,\ 1\rangle\right\|}\right|_{(x_0,y_0)} \stackrel{\text{def}}{=\!=} \text{the OUTWARD normal unit vector to } \mathcal{S} \text{ at the point } (x_0,y_0,g(x_0,y_0))\in\mathcal{S}$$

Note: $\left[\overrightarrow{T_x} \times \overrightarrow{T_y}\right](x_0,y_0)$ points *upwards* since its last coordinate is positive.

Φ is orientation-preserving

7.e. Surface Integrals of Scalar Fields

7.e.1. Given: $\underset{\mathbb{R}^2}{\overset{\cap}{D^2}} \xrightarrow[\text{onto}]{\Phi} \underset{\mathbb{R}^3}{\overset{\cap}{\mathcal{S}}} \xrightarrow{f} \mathbb{R}$

with: Φ is NICE

f is continuous on $\Phi(D^2)$

then: the INTEGRAL OF f OVER $\mathcal{S}$ is :

$$\iint_{\mathcal{S}} f\, dS \;=\; \iint_{D^2} f\left(\Phi\left(u,v\right)\right)\, \left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|\; dudv\ .$$

7.e.2. Key Idea: using notation from **5.2** (see text pg 431&442)

Form: a partition $\mathcal{R} = \{R_{ij} \,:\, R_{ij} \subset D^2\}$ of D^2 into rectangles

where: mesh $\mathcal{R} \approx 0$

and: R_{ij} has sides of length Δu_{ij} and Δv_{ij}

pick: a selection $\{c_{ij}\}$ where $c_{ij} \in R_{ij}$.

Then:

$$\begin{aligned}
\iint_{\mathcal{S}} f\, dS &= \iint_{D^2} f\left(\Phi\left(u,v\right)\right) \left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|\; dudv \\
&\approx \sum_{R_{ij}\in\mathcal{R}} \left[\, f(\Phi(c_{ij})) \left\|\overrightarrow{T_u}(c_{ij}) \times \overrightarrow{T_v}(c_{ij})\right\| \,\right] \Delta u_{ij}\, \Delta v_{ij} \\
&= \sum_{R_{ij}\in\mathcal{R}} f(\Phi(c_{ij})) \left[\, \left\|\overrightarrow{T_u}(c_{ij}) \times \overrightarrow{T_v}(c_{ij})\right\| \Delta u_{ij}\, \Delta v_{ij} \,\right] \\
&\overset{\textbf{7.d.2}}{\approx} \sum_{R_{ij}\in\mathcal{R}} f(\Phi(c_{ij}))\ [\text{ surface area of } \Phi(R_{ij})\ ]\ .
\end{aligned}$$

7.e.3. Examples: compare with **7.b.3**

▷ $\iint_{\mathcal{S}} 1\, dS$ = surface area of $\mathcal{S}$.

▷ If: $\mathcal{S}$ represents a surface in $\mathbb{R}^3$

with: $\rho\colon \mathbb{R}^3 \to \mathbb{R}$ the mass density (eg. grams/cm^2) of $\mathcal{S}$

then: the total mass of the $\mathcal{S}$ $= \iint_{\mathcal{S}} \rho\, dS$

and: the center of mass of $\mathcal{S}$ $= \left(\frac{\iint_{\mathcal{S}} x\rho\, dS}{\iint_{\mathcal{S}} \rho\, dS} \,,\, \frac{\iint_{\mathcal{S}} y\rho\, dS}{\iint_{\mathcal{S}} \rho\, dS} \,,\, \frac{\iint_{\mathcal{S}} z\rho\, dS}{\iint_{\mathcal{S}} \rho\, dS} \right)$.

▷ the AVERAGE VALUE of f on $\mathcal{S}$ is $\frac{\iint_{\mathcal{S}} f\, dS}{\iint_{\mathcal{S}} 1\, dS}$

eg: $f(x,y,z)$ is the temperature at the point (x,y,z)

so: average value of f on $\mathcal{S}$ = average temperature of f on $\mathcal{S}$.

7.g. Surface Integrals of Vector Fields

7.g.1. Given $\underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow[\text{onto}]{\Phi} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}} \xrightarrow{\vec{F}} \mathbb{R}^3$

where Φ is NICE and $\vec{F}$ is continuous on $\Phi(D^2)$

and $\mathcal{S}$ is oriented by Φ , i.e.

$$\frac{(\overrightarrow{T_u} \times \overrightarrow{T_v})}{\|\overrightarrow{T_u} \times \overrightarrow{T_v}\|} = \text{the OUTWARD unit normal vector to } \mathcal{S} \overset{\text{def}}{=} \vec{n} .$$

Then the SURFACE INTEGRAL OF $\vec{F}$ OVER $\mathcal{S}$ is :

$$\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} \overset{\text{def}}{=} \iint_{D^2} \left[\vec{F}\left(\Phi\left(u,v\right)\right) \bullet \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right) \right] dudv \overset{\textbf{7.e.1}}{=} \iint_{\mathcal{S}} \left[\vec{F} \bullet \vec{n}\right] dS .$$

7.g.2. In the set-up of **7.g.1** and **7.e.1**:

If $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$ is a $\begin{bmatrix} \text{orientation-preserving} \\ \text{orientation-reversing} \end{bmatrix}$ (re)-parmaterization of $\mathcal{S}$

then $\vec{n} = \left.\frac{(\overrightarrow{T_u}\times\overrightarrow{T_v})}{\|\overrightarrow{T_u}\times\overrightarrow{T_v}\|}\right|_{\text{for } \Phi} = \left.\frac{(\overrightarrow{T_u}\times\overrightarrow{T_v})}{\|\overrightarrow{T_u}\times\overrightarrow{T_v}\|}\right|_{\text{for } \Phi_1} = -\left.\frac{(\overrightarrow{T_u}\times\overrightarrow{T_v})}{\|\overrightarrow{T_u}\times\overrightarrow{T_v}\|}\right|_{\text{for } \Phi_2}$

so $\left.\overrightarrow{T_u} \times \overrightarrow{T_v}\right|_{\text{for } \Phi} = \left.\overrightarrow{T_u} \times \overrightarrow{T_v}\right|_{\text{for } \Phi_1} = -\left(\left.\overrightarrow{T_u} \times \overrightarrow{T_v}\right|_{\text{for } \Phi_2} \right)$

so $\iint_{\Phi} \vec{F} \bullet d\vec{S} = \iint_{\Phi_1} \vec{F} \bullet d\vec{S} = - \iint_{\Phi_2} \vec{F} \bullet d\vec{S}$

and $\iint_{\Phi} f\, dS = \iint_{\Phi_1} f\, dS = \iint_{\Phi_2} f\, dS$

7.g.3. Key Idea: using notation from **7.e.2** (see text pg 431 & 455)

So:

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} &\approx \sum_{R_{ij}\in\mathcal{R}} \left[\vec{F}(\Phi(c_{ij})) \bullet \left(\overrightarrow{T_u}(c_{ij}) \times \overrightarrow{T_v}(c_{ij})\right) \right] \Delta u_{ij}\, \Delta v_{ij} \\
&= \sum_{R_{ij}\in\mathcal{R}} \left[\vec{F}(\Phi(c_{ij})) \bullet \vec{n}(\Phi(c_{ij})) \right] \left(\left\| \overrightarrow{T_u}(c_{ij}) \times \overrightarrow{T_v}(c_{ij}) \right\| \Delta u_{ij}\, \Delta v_{ij} \right) \\
&\overset{\textbf{7.d.2}}{\approx} \sum_{R_{ij}\in\mathcal{R}} \left[\vec{F}(\Phi(c_{ij})) \bullet \vec{n}(\Phi(c_{ij})) \right] [\text{ surface area of } \Phi(R_{ij}) \text{ }] .
\end{aligned}$$

7.g.4. FLUX Recall **4.5b**

Have: a fluid flowing through a region $\mathbb{D}^3$ in $\mathbb{R}^3$ with:

- $\vec{v}$ - velocity vector field of the fluid $\left(\frac{\text{m}}{\text{sec}}\right)$
- δ - mass density scalar field of the fluid $\left(\frac{\text{gr}}{\text{m}^3}\right)$
- $\overrightarrow{F} \overset{\text{def}}{=} \delta\vec{v}$ - mass flow rate density of the fluid $\left(\frac{\text{gr}}{\text{sec}} \text{ per m}^2\right)$

a porous surface $\mathcal{S}$ sitting in $\mathbb{D}^3$:

through which the fluid flows without hindrance & with outward UNIT normal $\vec{n}$.

Def: **flux** of $\overrightarrow{F}$ across $\mathcal{S}$ $\overset{\text{def}}{=}$ net mass of fluid crossing $\mathcal{S}$ per unit time

$$= \frac{\left([\text{mass of fluid coming out of } \mathcal{S}] - [\text{mass of fluid going into } \mathcal{S}]\right) \text{ during } \Delta t}{\Delta t}$$

Claim: flux of $\overrightarrow{F}$ across $\mathcal{S}$ $= \iint_{\mathcal{S}} \left[\overrightarrow{F} \bullet \vec{n} \right] dS$

 new stuff

8.a. Summary of Ch 8 $\quad \vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$; $\operatorname{div} \vec{F} = \vec{\nabla} \bullet \vec{F}$; $\overrightarrow{\text{curl}}\, \vec{F} = \vec{\nabla} \times \vec{F}$

Example: $D \stackrel{\text{def}}{=} \{(x,y)\colon x^2+y^2 \leqslant 1\} \subset \mathbb{R}^2$ is a Green's Region

$\delta D = \{(x,y)\colon x^2+y^2 = 1\} \subset D$ is the boundary of D

$\delta D^+ = \langle \cos t, \sin t\rangle \quad 0 \leqslant t \leqslant 2\pi$ is a (counterclockwise) parameterization of δD

$\vec{n} = \langle x, y\rangle$ is the outward unit normal to δD .

8.a.1. Let: D be a Green's region

$\vec{F}(x,y) = \langle P(x,y)\, ,\, Q(x,y)\rangle$ be C^1 on D .

▶ Green's Theorem for $D \subset \mathbb{R}^2$

Then: $\int_{\delta D^+} \vec{F} \bullet d\vec{s} = \iint_D \left[\overrightarrow{\text{curl}}\, \vec{F} \bullet \vec{k}\right] dA$

$\qquad\qquad \| \qquad\qquad\qquad \|$

ie: $\int_{\delta D^+} P\,dx + Q\,dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$

⇝ so: $\int_{\delta D^+} \langle -y\, ,\, x\rangle \bullet d\vec{s} = 2$ (area of D) .

▶ Divergence Theorem for $D \subset \mathbb{R}^2$

Then: $\int_{\delta D^+} \left[\vec{F} \bullet \vec{n}\right] ds = \iint_D \left[\operatorname{div}\ \vec{F}\right] dA$.

Example: $\mathcal{S} \stackrel{\text{def}}{=} \{(x,y,z)\colon x^2+y^2+z^2 = 1 \text{ and } z \geqslant 0\} \subset \mathbb{R}^3$ is a Stokes' Surface

$\delta\mathcal{S} = \{(x,y,0)\colon x^2+y^2 = 1\} \subset \mathcal{S}$ is the boundary of $\mathcal{S}$

$\delta\mathcal{S}^+ = \langle \cos t, \sin t, 0\rangle \quad 0 \leqslant t \leqslant 2\pi$ is a (cc-wise) parameterization of $\delta\mathcal{S}$.

8.a.2. Stokes' Theorem for $\mathcal{S} \subset \mathbb{R}^3$

If: $\mathcal{S}$ is a Stokes' Surface

$\vec{F}\colon \mathcal{S} \to \mathbb{R}^3$ is C^1 on $\mathcal{S}$

then: $\int_{\delta\mathcal{S}^+} \vec{F} \bullet d\vec{s} = \iint_{\mathcal{S}} \left[\overrightarrow{\text{curl}}\, \vec{F}\right] \bullet d\vec{S}$.

Example: $\Omega \stackrel{\text{def}}{=}$ soup can (top & bottom included) + the soup is a Gauss' Region

$\delta\Omega =$ the soup can (top & bottom included) is the boundary of Ω .

8.a.3. Gauss' Divergence Theorem for $\Omega \subset \mathbb{R}^3$

If: Ω is a Gauss' Region

$\vec{F}\colon \Omega \to \mathbb{R}^3$ is C^1 on Ω

then: $\iint_{\delta\Omega} \vec{F} \bullet d\vec{S} = \iiint_{\Omega} \left[\operatorname{div}\ \vec{F}\right] dV$.

8.b. Definitions of Green's Region, Stokes' Surface, Gauss' Region   see **7.a** for path defs

8.b.1. Elementary Green's Region D and corresponding items:

$D \subset \mathbb{R}^2$ is an elementary region (**5.2**)

δD is the boundary of D (**6.2**)

$\delta D^+ \;=\; \langle x(t)\,,\, y(t)\rangle$ is a piecewise-smooth (oriented simple closed) path that parameterizes δD in the positive (ie. counterclockwise) direction

$\vec{n} \;=\; \dfrac{\langle\, y'(t)\,,\, -x'(t)\,\rangle}{\|\,\langle\, y'(t)\,,\, -x'(t)\,\rangle\,\|}$ is the outward unit normal vector to δD .

⇝ note that if you walk along δD^+, then D is to your left

⇝ a region that can be divided into a finite number of Elementary Green's Regions, with their shared δD's "oppositely oriented," is a Green's Region (see p. 471)

8.b.2. Elementary Stokes' Surface $\mathcal{S}$ and corresponding items:

Have: $\underset{\mathbb{R}^2}{\overset{D^2}{\cap}} \xrightarrow[\text{onto}]{\Phi} \underset{\mathbb{R}^3}{\overset{\mathcal{S}}{\cap}}$

where Φ is a NICE parameterization of the surface $\mathcal{S}$ (**7.d.3**) & D is an elem. Green's region

Φ is C^2 on (all of) D and 1-to-1 on (all of) D

$\mathcal{S}$ is smooth and oriented by Φ. (**7.f.1**)

So have: $\delta D^+ \overset{\mathbf{8.b.1}}{=\!=} \langle u(t)\,,\, v(t)\rangle$ is as above in **8.b.1** .

Let: $\delta S \overset{\text{def}}{=} \Phi(\delta D)$

$\delta S^+ \overset{\text{def}}{=} \Phi(\delta D^+)$.

So: $\delta\mathcal{S}$ is the *boundary* of $\mathcal{S}$

δS^+ is a (oriented simple closed) path that parameterizes δS .

also need: if you walk along δS^+, standing on the positive side of $\mathcal{S}$, then $\mathcal{S}$ is to your left .

Special Case$^+$ $\Phi(x,y) \;=\; (x, y, g(x,y))$ where $z = g(x,y)$ is C^2 and D is an elem. Green's region .

⇝ Think of $D \subset \mathbb{R}^2$ as a piece of lycra with a wire as its boundary and Φ places D down in $\mathbb{R}^3$ and then *gently* transforms it (1-to-1-ish) to $\mathcal{S}$ so that $\mathcal{S}$'s outward unit normals vary continously.

⇝ a surface that can be divided into a finite number of Elementary Stokes' Surfaces, with their shared δS's "oppositely oriented," is a Stokes' Surface

8.b.3. Gauss' Region Ω and corresponding items:

$\Omega \subset \mathbb{R}^3$ is a simple region (**5.3**) , thus has *top, bottom,* and *possibly sides*

$\delta\Omega \subset \mathbb{R}^3$ is the boundary of Ω (**6.2**)

$\delta\Omega$ is oriented by the outward normal vectors

$\delta\Omega$ is piecewise NICE (**7.d.3**) .

STOKE'S SURFACES

8.d.1. Let: $D^2 \;=\; \{(x,y)\colon x^2+y^2 \leqslant 1\}$

$\mathcal{S} \;=\; \{(x,y,z)\colon x^2+y^2+z^2=1 \text{ and } z \geqslant 0\}$.

Define $\Phi\colon D^2 \xrightarrow{\text{onto}} \mathcal{S}$ by

$$\Phi(x,y) \;=\; \left(\frac{2x}{x^2+y^2+1} \,,\, \frac{2y}{x^2+y^2+1} \,,\, \frac{1-x^2-y^2}{x^2+y^2+1} \right) .$$

Claim: $\Phi(x,y)$ is the pt. of intersection of $\mathcal{S}$ & the line thru $(x,y,0)$ & $(0,0,-1)$.

Note

$$\overrightarrow{T_x} \times \overrightarrow{T_y} \;=\; \left\langle \frac{8x}{(x^2+y^2+1)^3} \,,\, \frac{8y}{(x^2+y^2+1)^3} \,,\, \frac{4\left(1-x^2-y^2\right)}{(x^2+y^2+1)^3} \right\rangle .$$

Thus $\Phi\colon D^2 \xrightarrow{\text{onto}} \mathcal{S}$ realizes the upper-hemisphere as a Stoke's surface.

Also: $\delta D^+(t) \;=\; \langle \cos t \,,\, \sin t \rangle$

$\delta S^+(t) \overset{\text{def}}{=\!=} \Phi(\delta D^+)(t) \;=\; \langle \cos t \,,\, \sin t \,,\, 0 \rangle$

So we now see that the sphere is a Stoke's Surface (why?). What's $\delta S^+(t)$?

8.d.2. Let: $0 < h = \text{height}$ and $0 < r = \text{radius}$.

$D^2 \;=\; \{(x,y)\colon -r \leqslant x \leqslant r \text{ and } 0 \leqslant y \leqslant h\}$

$\mathcal{S} \;=\; \{(x,y,z)\colon x^2+z^2=r^2 \text{ and } z \geqslant 0 \text{ and } 0 \leqslant y \leqslant h\}$.

Define $\Phi\colon D^2 \xrightarrow{\text{onto}} \mathcal{S}$ by

$$\Phi(x,y) \;=\; \left(\frac{2xr^2}{r^2+x^2} \,,\, y \,,\, \frac{r(r^2-x^2)}{r^2+x^2} \right) .$$

Claim: $\Phi(x,y)$ is the pt. of intersection of $\mathcal{S}$ & the line thru $(x,y,0)$ & $(0,y,-r)$.

Note

$$\overrightarrow{T_x} \times \overrightarrow{T_y} \;=\; \left\langle \frac{4xr^4}{(r^2+x^2)^2} \,,\, 0 \,,\, \frac{2r^2\left(r^2-x^2\right)}{(r^2+x^2)^2} \right\rangle .$$

Thus $\Phi\colon D^2 \xrightarrow{\text{onto}} \mathcal{S}$ realizes a semi-cylinder as a Stoke's surface.

Also: $\delta D^+(t) \;= ?$

$\delta S^+(t) \overset{\text{def}}{=\!=} \Phi(\delta D^+)(t) \;= ?$

So we now see that a cylinder is a Stoke's Surface (why?). What's $\delta S^+(t)$?

GAUSS' THEOREM

The key idea behind the proof of Gauss' Theorem is **4.5b**.

To see this, first recall **4.5b**; if you wish, let $\delta = 1$ so the flux is just the net volume of fluid crossing a surface per unit time.

8.e.. Have: a fluid flowing through a region $\mathbb{D}^3$ in $\mathbb{R}^3$ with:

$\vec{v}$ - velocity vector field of the fluid $\left(\frac{\text{m}}{\text{sec}}\right)$

δ - mass density scalar field of the fluid $\left(\frac{\text{gr}}{\text{m}^3}\right)$

$\overrightarrow{F} \overset{\text{def}}{=\!=} \delta\vec{v}$ - mass flow rate density of the fluid $\left(\frac{\text{gr}}{\text{sec}} \text{ per m}^2\right)$

a porous GAUSS REGION Ω sitting in $\mathbb{D}^3$ so that:

- the fluid flows through Ω without hindrance
- $\delta\Omega$ is a surface with outward UNIT normal $\vec{n}$.

8.e.1. Divide $\delta\Omega$ into lots of small (almost flat) patches $\mathcal{S}_i$.

4.5.b (i) $\Longrightarrow$ flux of $\vec{F}$ across $\mathcal{S}_i \approx \left(\vec{F} \bullet \vec{n}\right)$ (area $\mathcal{S}_i$) .

Calculus $\Longrightarrow$ flux of $\vec{F}$ across $\delta\Omega = \iint_{\delta\Omega} \left(\vec{F} \bullet \vec{n}\right) dS = \iint_{\delta\Omega} \vec{F} \bullet d\vec{S}$.

8.e.2. Divide Ω into lots of small (solid) rectangular boxes R_i.

4.5.b (ii) $\Longrightarrow$ flux of $\vec{F}$ across $\delta R_i \approx \left(\text{div } \vec{F}\right)$ (vol R_i) .

Calculus $\Longrightarrow$ flux of $\vec{F}$ across $\delta\Omega = \iiint_{\Omega} \left[\text{ div } \vec{F} \right] dV$.

8.c. Conservative Fields

8.c.1. <u>Theorem</u> (compare with **4.4**)

If $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ is a C^1 vector field

where $\mathbb{D}^3 =$ all of $\mathbb{R}^3$ expect for possibly a finite number of points,

then the following are equivalent:

(1) $\vec{F} = \overrightarrow{\nabla} f$ for some scalar function $f\colon \mathbb{D}^3 \to \mathbb{R}$

(2) $\overrightarrow{\text{curl}}\,\vec{F} = \vec{0}$

(3) $\int_{\vec{c}} \vec{F} \bullet d\vec{s} = 0$ for *each* oriented simple closed curve $\vec{c}$

(4) $\int_{\vec{c}_1} \vec{F} \bullet d\vec{s} = \int_{\vec{c}_2} \vec{F} \bullet d\vec{s}$ for *each* pair of oriented simple curve $\vec{c}_1$ & $\vec{c}_2$ that both start at the same point and both end at the same point.

If $\vec{F}$ satisfies one (and hence all) of the above conditions,

then: $\vec{F}$ is a GRADIENT VECTOR FIELD

$\vec{F}$ is a CONSERVATIVE VECTOR FIELD

f is the POTENTIAL of $\vec{F}$

$$f(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

where $\vec{F} = \langle F_1, F_2, F_3 \rangle$.

7.c.7. Given: $\vec{c}\colon [a,b] \to \mathbb{R}^3$ with $\vec{c}$ piecewise C^1

$f\colon \mathbb{R}^3 \to \mathbb{R}$ with $f \in C^2$

so $\overrightarrow{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $\overrightarrow{\nabla} f \in C^1$

then: $\int_{\vec{c}} \overrightarrow{\nabla} f \bullet d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$.