

MARK BOX		
PROBLEM	POINTS	
1 a-c	16	
2 a-e	16	
3	16	
4 a-e	16	
5	16	
6	16	
bonus	4	
Total	100	

Math 550A      Prof. Girardi

Spring 98      Final Exam

5/2/98

NAME: \_\_\_\_\_

**INSTRUCTIONS:**

1. To receive credit you must:
  - a. WORK IN A LOGICAL FASHION, SHOW ALL YOUR WORK, INDICATE YOUR REASONING
  - b. when applicable put your answer on/in the line/box provided
  - c. if no such line/box is provided, then box your answer.
2. The MARK BOX indicates the problems along with their points.  
Check that your copy of the exam has all of the problems.
3. As indicated on the syllabus:
  - a. allowed is a calculator
  - b. allowed are the class handouts: table of integrals, calculus formula sheet, and informal summary (along with your personal scribbles on them)
  - c. not allowed are other notes and books.
4. This exam covers (from *Vector Calculus* by Marsden & Tromba, 4<sup>th</sup> ed.) :  
Chs. 1, 2, § 3.1, Chs. 4, 5, 6, 7, 8 .

The sources of the questions on this exam are:

- (1) Quiz problem § 2.6 # 16.
- (2) 1997 Final Exam # 2.
- (3) An exercise from another Vector Calculus book.
- (4) Motivated by an example from the last day of class.
- (5) Homework: Chapter 8 Review # 3.
- (6) Homework: Chapter 8 Review # 5.

\_\_\_ I am using a fancy calculator so often my “work” is not shown on items such as computing cross products.

\_\_\_ I am not using a fancy calculator so my work is shown.

1. Captain Ralph is in trouble near the sunny side of Mercury. The temperature of the ship's hull when he is at location  $(x, y, z)$  will be given by

$$T(x, y, z) = \exp(-x^2 - 2y^2 - 3z^2)$$

where  $x, y,$  and  $z$  are measured in meters. He is currently at  $(1, 1, 1)$  and is moving with constant speed of  $e^8$  meters per second.

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- 1a. In order to decrease the temperature most rapidly, Captain Ralph should move in the direction of the vector: \_\_\_\_\_ .

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- 1b. If Captain Ralph proceeds in the direction suggested in (1a), the temperature will changing at a rate of \_\_\_\_\_ degrees per second.  
(Be careful ... is your answer positive or negative?)

continued →

- 1c.** Unfortunately, the metal of the hull will crack if cooled at a rate greater than  $\sqrt{(14e^2)}$  degrees per second. Describe the set of possible directions in which he may proceed to bring the temperature down at no more than that rate.

2. Let  $m$ ,  $r_1$  and  $r_2$  be fixed arbitrary positive constants with  $r_1 < r_2$ .

Consider the frustum  $\mathcal{S}$  that is the surface of the cone:

$$z = m \sqrt{x^2 + y^2}$$

that sits between the planes  $z = mr_1$  and  $z = mr_2$ .

So  $\mathcal{S}$  is just the side part and does NOT include the top nor bottom "lids".

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2a. Make a rough sketch of the  $\mathcal{S}$ , labeling the two points on  $\mathcal{S}$  of the form  $(0, ?, mr_1)$  and  $(0, ?, mr_2)$ , indicating what ? are.

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2b. A parameterization of  $\mathcal{S}$  is:

$$\Phi(r, \theta) = \underline{\hspace{15cm}},$$

over the domain

$$D = \{(r, \theta) : \underline{\hspace{2cm}} \leq r \leq \underline{\hspace{2cm}} \quad \text{and} \quad \underline{\hspace{2cm}} \leq \theta \leq \underline{\hspace{2cm}}\}.$$

continued  $\rightarrow$

**2c.** A (not necessarily normalized) normal vector to  $\mathcal{S}$  is:

$$\vec{T}_r \times \vec{T}_\theta = \underline{\hspace{15cm}},$$

and this vector has length :  $\underline{\hspace{15cm}}$ .

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**2d.** The surface area of  $\mathcal{S}$  can be expressed as the following REIMANN integral over  $D$ :

$$\text{SA} = \underline{\hspace{15cm}}.$$

**2e.** Integrating the integral in **2d** gives that the surface area of  $\mathcal{S}$  is:

$$\text{SA} = \underline{\hspace{15cm}}.$$

**3. Green's Theorem:** Let WORK be the work of the force:

$$\vec{F}(x, y) = \langle x, 3x + y^2 \exp(\sqrt{4y^{10} + 17}) \rangle$$

done on a particle that moves along the straight line segments from  $(0, 0)$  to  $(1, 2)$ , then from  $(1, 2)$  to  $(3, 0)$ , then from  $(3, 0)$  back to  $(0, 0)$ .

**3a.** WORK can be expressed as the following line integral:

WORK = \_\_\_\_\_ .  
along the curve  $\vec{c}$  that is indicated in my sketch below. You do not need to specifically parameterize  $\vec{c}$ , just draw it and indicate its orientation.

**3b.** WORK can be expressed as the following number:

WORK = \_\_\_\_\_ .

4. Stokes' Theorem: Let  $\mathcal{S}$  be the surface of the paraboloid  $z = x^2 + y^2$  for  $0 \leq z \leq 1$ , with the upward orientation (so the  $z$ -coordinate of the unit normal is positive).  $\mathcal{S}$  does *not* include the *lid* that sits in the  $z = 1$  plane. Let  $\vec{F}$  be:

$$\vec{F}(x, y, z) = \langle z^4 e^{z^{17}+5}, \quad 3x, \quad x \tan(z^{17}) + y^2 \exp(\sqrt{4y^{10} + 17z^5 + 9}) \rangle.$$


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- 4a. Make a rough sketch of  $\mathcal{S}$ , indicating its orientation and its Stokes' boundary.

- 4b. A parameterization  $\Phi$  that realizes  $\mathcal{S}$  as a Stokes' surface is:

$$\Phi(x, y) = \langle x, y, \underline{\hspace{2cm}} \rangle,$$

for  $(x, y)$  in  $D$  where:

$D$  is \_\_\_\_\_.

- 4c. The corresponding (oriented) boundary  $\delta\mathcal{S}^+$  of  $\mathcal{S}$  is the curve:

$$\vec{c}(t) = \langle \underline{\hspace{2cm}}, \underline{\hspace{2cm}}, \underline{\hspace{2cm}} \rangle$$

for

$$\underline{\hspace{1cm}} \leq t \leq \underline{\hspace{1cm}}.$$

continued  $\rightarrow$

**4d.** As a hint to 4e, note that  $\overrightarrow{\text{curl } \vec{F}} \cdot \vec{k} = \underline{\hspace{15em}}$  .

**4e.**  $\int \int_{\mathcal{S}} \left[ \overrightarrow{\text{curl } \vec{F}} \right] \cdot d\vec{S} = \underline{\hspace{15em}}$  .

HINT: brute force will not work . . . you need to be clever.

5. Gauss' Theorem: Let  $\mathcal{S}$  be the surface of the unit cube and  $\vec{F}$  be

$$\vec{F}(x, y, z) = \langle x^2y, z^8, -2xyz \rangle .$$

Let FLUX be the flux of  $\vec{F}$  across  $\mathcal{S}$ .

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5a. FLUX can be expressed as the following SURFACE INTEGRAL:

FLUX = \_\_\_\_\_ .

5b.  $\operatorname{div} \vec{F} =$  \_\_\_\_\_ .

5c. FLUX is the following number (explain your answer):

FLUX = \_\_\_\_\_ .

**6. Conservative Fields:**

A puffo is running along a path given by

$$\vec{c}(t) = \langle \cos^3 t, \sin^3 t, 0 \rangle,$$

for  $0 \leq t \leq \frac{\pi}{2}$ , while being subjected to a force field

$$\vec{F}(x, y, z) = \langle x^3 - 2xy^3, -3x^2y^2, 0 \rangle.$$

Let WORK be the work done by  $\vec{F}$  on the puffo as he moves along  $\vec{c}$ .

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**6a.** WORK can be expressed as the following line integral:

WORK = \_\_\_\_\_ .

**6b.**  $\overrightarrow{\text{curl}} \vec{F} =$  \_\_\_\_\_ .

**6c.** Is  $\vec{F}$  conservative? \_\_\_\_\_ (yes/no). Briefly explain your answer.

**6d.** WORK can be expressed as the following number:

WORK = \_\_\_\_\_ .