MARK BOX		
PROBLEM	POINTS	
1	20	
2	20	
3	20	
4	20	
5 a–b	20	
Total	100	

Math 550A Prof. Girardi

Spring 98 Exam 1

2/14/97

NAME: ______________________________

INSTRUCTIONS:

1. To receive credit you must:
   a. WORK IN A LOGICAL FASHION, SHOW ALL YOUR WORK, INDICATE YOUR REASONING
   b. when applicable put your answer on/in the line/box provided
   c. if no such line/box is provided, then box your answer.
2. The MARK BOX indicates the problems along with their points.
   Check that your copy of the exam has all of the problems.
3. As indicated on the syballus:
   a. allowed is a calculator
   b. allowed are the class handouts: table of integrals, calculus formula sheet, and informal summary (along with your personal scribbles on them)
   c. not allowed are other notes and books.
4. This exam covers (from *Vector Calculus* by Marsden & Tromba, 4^{th}ed.) :
   Ch. 1, Ch. 2, § 3.1, § 4.1, § 4.2 .

The sources of the questions on this exam are:

(1) my typical first exam first question,
(2) a standard question,
(3) similar to: 1993 Exam 1 # 2,
(4) similar to: 1993 Exam 1 # 5,
(5) something new.

___ I am using a fancy calculator so often my "work" is not shown on items such as computing cross poducts.

___ I am not using a fancy calculator so my work is shown.

1. Let $\vec{A} = \langle 1, 2, -4\rangle$ and $\vec{B} = \langle 3, 1, 2\rangle$. Let θ be the angle between $\vec{A}$ and $\vec{B}$. Let $\vec{A} = \vec{A}_{\|} + \vec{A}_{\perp}$ where $\vec{A}_{\|}$ is parallel to $\vec{B}$ and $\vec{A}_{\perp}$ is perpendicular to $\vec{B}$. Find:

$\left\|\vec{A}\right\| =$	$\cos\theta =$
$\left\|\vec{B}\right\| =$	is $0 \leq \theta \leq \frac{\Pi}{2}$ or $\frac{\Pi}{2} < \theta \leq \Pi$?____
$\vec{A} \bullet \vec{B} =$	$\vec{A}_{\|} =$
$\vec{A} \times \vec{B} =$	$\vec{A}_{\perp} =$

2. A parameterization (in the form of $\vec{R}(t) = \langle ?, ?, ? \rangle$) of the line that passes through the point $(1, 2, 3)$ and is perpendicular to the vectors $\vec{A} = \langle 1, 2, 1 \rangle$ and $\vec{B} = \langle 1, 0, -3 \rangle$ is:

____ .

3. Consider the scalar field $f(x, y, z) = 3x - y^2 - ye^z$ along with the point $P = (2, 1, 0)$. Find:

3a. $\overrightarrow{\nabla} f(x, y, z) = \langle$ ________________ , ________________ , ________________ $\rangle$

$\overrightarrow{\nabla} f(2, 1, 0) = \langle$ ________________ , ________________ , ________________ $\rangle$

3b. The directional derivative of f at the point P in the direction of $\vec{u} = \langle 1, 0, 1\rangle$ is:

$D_{\vec{u}} f(2, 1, 0) =$ ____ .

3c. At P, the function f increases most rapidly in the direction of $\langle$ ___ , ___ , ___ $\rangle$.

3d. An equation (in the form of $ax + by + cz = d$) of the plane tangent to the surfacce $f(x, y, z) = 4$ at the point P is:

____ .

4. A puffo is running along the branch of the curve $xy = 1$ that lies in the first quadrant in such a way that its abscissa (ie. the x coordinate) is increasing in time. The puffo's speed (i.e. SCALAR velocity) is a constant 4 feet per minute. Express his velocity vector as a function of the abscissa (ie. as a function of x and not time t).

ANSWER: $\vec{v}(x) \; = <$ ________________________ , ________________________ $>$

5a. Fix a point $A = (a_1, a_2) \in \mathbb{R}^2$. For any point $P = (x, y) \in \mathbb{R}^2$, let

$$d_A(P) \;=\; \text{the distance between } A \text{ and } P$$

that is,

$$d_A(x,y) \;=\; \sqrt{(x-a_1)^2 \;+\; (y-a_2)^2}$$

Show that:

$$\vec{\nabla} d_A(P) \quad \text{is the UNIT vecter in the direction of } \overrightarrow{AP}$$

that is, show that:

$$\vec{\nabla} d_A(P) \;=\; \frac{\overrightarrow{AP}}{\left\|\overrightarrow{AP}\right\|}\,.$$

<u>HINT</u>: WAY # 1: First compute $\vec{\nabla} d_A(P) \overset{\text{def}}{=} \langle \frac{\partial d_A(x,y)}{\partial x}, \frac{\partial d_A(x,y)}{\partial y} \rangle$ and then algebraically manilulation it to make it look like $\frac{\overrightarrow{AP}}{\|\overrightarrow{AP}\|}$. WAY # 2: Think what the direction and length of $\vec{\nabla} d_A(P)$ mean geometrically and EXPLAIN why $\vec{\nabla} d_A(P)$ has to be what it is.

<u>NOTE</u>: this implies that $\vec{\nabla} d_A(P)$ is the UNIT vector in the direction of $\overrightarrow{AP}$.

5b. Now fix two points: $A = (a_1, a_2)$ and $B = (b_1, b_2)$. For any point $P = (x, y)$, let:

$$f(P) \;=\; [\text{the distance between } A \text{ and } P] \;+\; [\text{the distance between } B \text{ and } P]\;,$$

that is:

$$f(x,y) \;=\; d_A(x,y) \;+\; d_B(x,y)\;.$$

Fix $c > 0$ and consider the level set of f of value c, which is:

$$\mathcal{S} \overset{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 \colon f(x,y) = c\},$$

that is

$$\mathcal{S} \;=\; \{P \in \mathbb{R}^2 \colon d_A(P) \;+\; d_B(P) = c\}.$$

Since an ellipse is the set of all points P the sum of whose distances from two fixed points A and B (called the foci) is constant, we see that $\mathcal{S}$ is an ellipse.

Now fix a point P on $\mathcal{S}$ and let

$$\begin{gathered}\overrightarrow{T} \text{ be a UNIT tangent to } \mathcal{S} \text{ at } P\\ \alpha \text{ be the angle between } \overrightarrow{AP} \text{ and } \overrightarrow{T}\\ \beta \text{ be the angle between } \overrightarrow{BP} \text{ and } -\overrightarrow{T}\quad.\end{gathered}$$

Show that

$$\alpha = \beta\;.$$

<u>NOTE</u>: this problem has a physical interpretation; namely, light rays (or sound waves) originating at focus A will reflect from the ellipse to focus B.

<u>HINT</u>: Geometrically, what can you say about $\overrightarrow{\nabla} f$ in relation to $\mathcal{S}$? So what can say about $\overrightarrow{\nabla} f(P) \bullet \overrightarrow{T}$? Informal Summary **4.3** #1 and part (5a) should help to find $\overrightarrow{\nabla} f(P)$ and the Informal Summary **1.4** should then help with $\overrightarrow{\nabla} f(P) \bullet \overrightarrow{T}$.

<u>WARNING</u>: work your solution out on scratch paper and then write it neatly and logically on the next page in such a way that a fellow student could understand your explanation. No credit will be given for math symbols randomly thrown down on the page.