1.6. Section 1.6: Proofs Involving Quantifiers.

1.6.1. **Remark.** Many theorems in mathematics are quantified sentences, even though the quantifier may not actually appear in the statement. For example, proving "If x is an odd integer, then $x+1$ is even," as we did in Section 1.4, acually involves a quantifier, since the sentence has the symbolic translation

$$(\forall x \in \mathbb{Z})\ (x \text{ is odd } \Rightarrow\ x+1 \text{ is even}) .$$

In Sections 1.4 and 1.5, in order to concentrate on the basic proof forms, we suppressed the importance of quantifiers by viewing the variable as a *fixed* object; thus, our proofs of quantified statements in Sections 1.4 and 1.5 were not correct. In this section, we will learn how to *correctly* prove quantified sentences.

$\boxed{\forall}$

1.6.2. **Proof of** $(\forall x)P(x)$ (Direct Proof).

Let x be an arbitary object in the universe.
(or ... Fix an arbitary element x from the universe.)
$\vdots$
Hence, $P(x)$ is true .
Since x is arbitrary, $(\forall x)P(x)$ is true.

1.6.3. **Proof of** $(\forall x)P(x)$ (Proof by Contradiction).

Assume $\sim [\ (\forall x)P(x)\ ]$ is true .
Then $(\exists x)[\ \sim P(x)\ ]$ is true .
Let t be an object such that $\sim P(t)$ is true .
$\vdots$
Therefore $Q \wedge \sim Q$ is true , a contradiction.
Hence, $\sim [\ (\forall x)P(x)\ ]$ is false;
so its denial $(\forall x)P(x)$ is true.

$\boxed{\exists}$

1.6.4. **Proof of** $(\exists x)P(x)$ **(Constructive (direct) Proof).**

Specifically describe some object x in the universe that makes $P(x)$ true.

1.6.5. **Proof of** $(\exists x)P(x)$ **(Nonconstructive (indirect) Proof).**

Show that there must be some object x in the universe that makes $P(x)$ true,
without ever actually producing the object.

1.6.6. **Proof of** $(\exists x)P(x)$ **(Proof by Contradiction).**

Assume $\sim [\ (\exists x)P(x)\ ]$ is true .
Thus $(\forall x)[\ \sim P(x)\ ]$ is true ..
$\vdots$
Therefore $Q \wedge \sim Q$ is true , a contradiction.
Hence, $\sim (\exists x)P(x)$ is false ;
so $(\exists x)P(x)$ is true.

$\boxed{\exists\,!}$

1.6.7. **Proof of** $(\exists!x)P(x)$.

Do in two parts:

1st *Existence*:

Prove that $(\exists x)P(x)$ is true by any method.

2nd *Uniqueness*:

Assume that t_1 and t_2 are objects in the universe such that $P(t_1)$ and $P(t_2)$ are true.

$\vdots$

Therefore, $t_1 = t_2$.

We conclude that $(\exists!x)P(x)$.

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counterexample

1.6.8. **Definition.** A *counterexample* to $(\forall x)P(x)$ is any object t in the universe for which $P(t)$ is false.

1.6.9. **Remark** (Some helpful **TRUE** propositions.).

⋆ 1.	$(\forall x)(\forall y)P(x,y)$	$\Leftrightarrow$	$(\forall y)(\forall x)P(x,y)$	
⋆ 2.	$(\exists x)(\exists y)P(x,y)$	$\Leftrightarrow$	$(\exists y)(\exists x)P(x,y)$	
⋆ 3.	$(\forall x)\,[\,P(x)\wedge Q(x)\,]$	$\Leftrightarrow$	$[\,(\forall x)P(x)\wedge(\forall x)Q(x)\,]$	
⋆ 4.	$[\,(\forall x)P(x)\vee(\forall x)Q(x)\,]$	$\Rightarrow$	$(\forall x)\,[\,P(x)\vee Q(x)\,]$	
⋆ 5.	$(\forall x)\,[\,P(x)\Rightarrow Q(x)\,]$	$\Rightarrow$	$[\,(\forall x)P(x)\Rightarrow(\forall x)Q(x)\,]$	
⋆ 6.	$(\exists x)(\forall y)P(x,y)$	$\Rightarrow$	$(\forall y)(\exists x)P(x,y)$	

1.6.10. **Remark** (Some **FALSE** propositions that are common errors.).

⋆ 7.	$(\exists x)P(x)$	$\Rightarrow$	$(\forall x)P(x)$	
⋆ 8.	$(\forall x)\,[\,P(x)\vee Q(x)\,]$	$\Rightarrow$	$[\,(\forall x)P(x)\vee(\forall x)Q(x)\,]$	converse of 4
⋆ 9.	$[\,(\forall x)P(x)\Rightarrow(\forall x)Q(x)\,]$	$\Rightarrow$	$(\forall x)\,[\,P(x)\Rightarrow Q(x)\,]$	converse of 5
⋆ 10.	$(\forall y)(\exists x)P(x,y)$	$\Rightarrow$	$(\exists x)(\forall y)P(x,y)$	converse of 6

- To help remember 7., 8., and 9., let $U=\mathbb{N}$, $P(x)$: x is even, $Q(x)$: x is odd.
- To help remember 10., let $U=\mathbb{R}^2$, $P(x,y)$: $x<y$.
